Efficient assembly of high order continuous and discontinuous finite element operators

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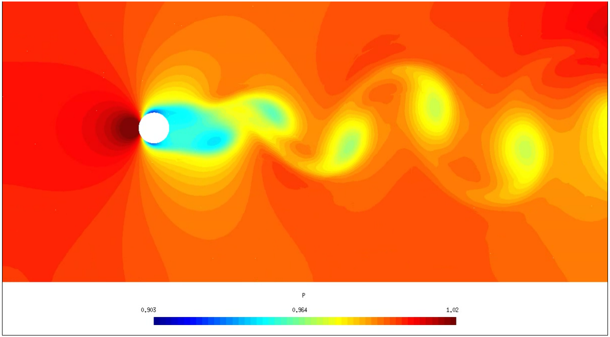
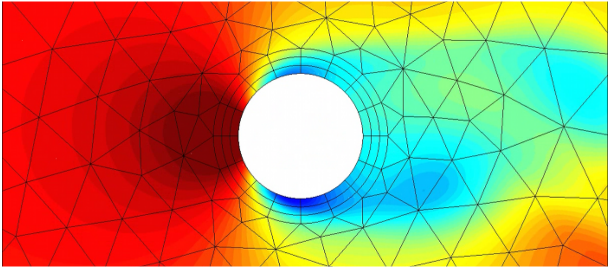
High order curved mixed meshes





Formulation

Continuous equation

$$0 = \nabla \cdot \mathbf{f}(u, \nabla u) + s(u, \nabla u)$$

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Discrete formulation

$$u|_{\Omega_e} = \sum_j u_j^e \phi_j$$

$$0 = \underbrace{\int_{\Omega_e} -\mathbf{f}(u, \nabla u) \cdot \nabla \phi_i ds}_{F_i^e} + \underbrace{\int_{\partial \Omega_e} q(u^d, \nabla u^d) \phi_i dl}_{Q_i^e}$$

$$+ \underbrace{\int_{\Omega_e} s(u, \nabla u) \phi_i ds}_{S_i^e}$$

$$i, j = 1, \ldots, N \ ; \ e = 1, \ldots, N_E \ ; \ d = 1, 2$$

Efficient numerical spatial integration

$$S_i^e = \int_{\Omega_e} s(u)\phi_i ds \simeq \sum_{\xi} s^{e\xi}\phi_i^{\xi} w^{\xi} J^{e\xi} \qquad \forall_i \forall_e$$

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1. evaluate the unknown field at the integration points
   $u^{e\xi} = \sum_i u_i^e \phi_i^{\xi} \quad \forall_e \forall_{\xi}$

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   $s^{e\xi} J^{e\xi} = s(u^{e\xi}) J^{e\xi} \ \ \forall_e \forall_{\xi}$

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3. perform the integration on the parent element
   $S_i^e \simeq \sum_{\xi} s^{e\xi}(u^{e\xi}) \phi_i^{\xi} w^{\xi} J^{e\xi} \quad \forall_e \forall_i$

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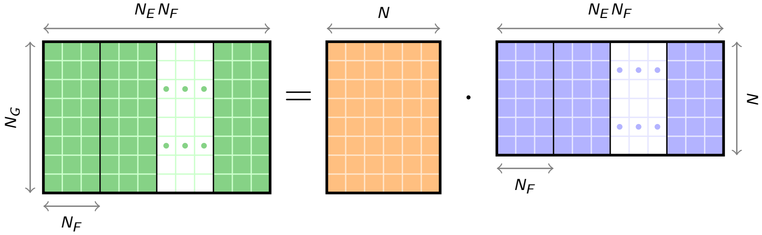
In high order, operations 1 and 3 are expensive
$\Rightarrow$ perform them as BLAS products.

Evaluate the fields at integration points $\mathcal{O}(N_E N_G N)$

Recast costly operations as BLAS3 products

$$u_k^{e\xi} = \sum_i \underbrace{\phi_i^\xi}_{A[\xi][i]} \underbrace{u_{ki}^e}_{B[i][ek]} \quad \forall_e \forall_\xi$$

$$\mathbf{U} = \mathbf{A} \cdot \mathbf{B}$$

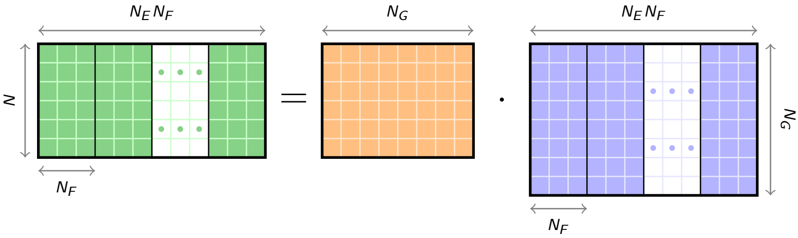


Integrate over the parent element $\mathcal{O}(N_E N_G N)$

Recast costly operations as BLAS3 products

$$S^e_{ki} \simeq \sum_{\xi} \underbrace{\phi_i^{\xi} w^{\xi}}_{A[i][\xi]} \underbrace{s_k^{e\xi} J^{e\xi}}_{B[\xi][ek]} \quad \forall_i \forall_e$$

$$\mathbf{S} = \mathbf{A} \cdot \mathbf{B}$$



Implicit : Evaluate the Jacobian matrix

$$F^e_{ki} = \int_{\Omega_e} -\mathbf{f}(u, \nabla u) \cdot \nabla \phi_i ds$$

$$\frac{dF^e_{ki}}{du^e_{lj}} = -\int_{\Omega e} \left[\frac{\partial \mathbf{f}_k}{\partial u_l} \frac{\partial u_l}{\partial u^e_{lj}} + \sum_b \frac{\partial \mathbf{f}_k}{\partial u_{l,x_b}} \frac{\partial u_{l,x_b}}{\partial u^e_{lj}} \right] \cdot \nabla \phi_i ds$$

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$$\simeq -\sum_{\xi a} \left[\frac{\partial f^{e\xi}_{ka}}{\partial u_l} \phi^\xi_j + \sum_b \frac{\partial f^{e\xi}_{ka}}{\partial u_{l,x_b}} \sum_\beta \phi^\xi_{j,\beta} Y^{e\xi}_{b\beta} \right] \sum_\alpha \phi^\xi_{i,\alpha} Y^{e\xi}_{a\alpha} w^\xi J^{e\xi}$$

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$$= -\sum_{\xi\alpha} \underbrace{w^\xi \phi^\xi_{i,\alpha} \phi^\xi_j}_{W_1[ij][\xi\alpha]} \underbrace{\sum_a \frac{\partial f^{e\xi}_{ka}}{\partial u^e_l} Y^{e\xi}_{a\alpha} J^{e\xi}}_{A_1[\xi\alpha][ekl]} - \sum_{\xi\alpha\beta} \underbrace{w^\xi \phi^\xi_{i,\alpha} \phi^\xi_{j,\beta}}_{W_2[ij][\xi\alpha\beta]} \underbrace{\sum_{ab} \frac{\partial f^{e\xi}_{ka}}{\partial u^e_{l,x_b}} Y^{e\xi}_{a\alpha} Y^{e\xi}_{b\beta} J^{e\xi}}_{A_2[\xi\alpha\beta][ekl]}$$

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$$F^e_{ki} = \int_{\Omega_e} -\mathbf{f}(u, \nabla u) \cdot \nabla \phi_i ds$$

$$\frac{dF^e_{ki}}{du^e_{lj}} = -\int_{\Omega e} \left[\frac{\partial \mathbf{f}_k}{\partial u_l} \frac{\partial u_l}{\partial u^e_{lj}} + \sum_b \frac{\partial \mathbf{f}_k}{\partial u_{l,x_b}} \frac{\partial u_{l,x_b}}{\partial u^e_{lj}} \right] \cdot \nabla \phi_i ds$$

$$\simeq -\sum_{\xi a} \left[\frac{\partial f^{e\xi}_{ka}}{\partial u_l} \phi^\xi_j + \sum_b \frac{\partial f^{e\xi}_{ka}}{\partial u_{l,x_b}} \sum_\beta \phi^\xi_{j,\beta} Y^{e\xi}_{b\beta} \right] \sum_\alpha \phi^\xi_{i,\alpha} Y^{e\xi}_{a\alpha} w^\xi J^{e\xi}$$

$$= -\sum_{\xi\alpha} \underbrace{w^\xi \phi^\xi_{i,\alpha} \phi^\xi_j}_{W_1[ij][\xi\alpha]} \underbrace{\sum_a \frac{\partial f^{e\xi}_{ka}}{\partial u^e_l} Y^{e\xi}_{a\alpha} J^{e\xi}}_{A_1[\xi\alpha][ekl]} - \sum_{\xi\alpha\beta} \underbrace{w^\xi \phi^\xi_{i,\alpha} \phi^\xi_{j,\beta}}_{W_2[ij][\xi\alpha\beta]} \underbrace{\sum_{ab} \frac{\partial f^{e\xi}_{ka}}{\partial u^e_{l,x_b}} Y^{e\xi}_{a\alpha} Y^{e\xi}_{b\beta} J^{e\xi}}_{A_2[\xi\alpha\beta][ekl]}$$

- Four large *BLAS3* matrix-matrix products
  $\mathbf{K} \stackrel{\text{def}}{=} \mathbf{W}_1 \cdot \mathbf{A}_1 + \mathbf{W}_2 \cdot \mathbf{A}_2 + \mathbf{W}_3 \cdot \mathbf{A}_3 + \mathbf{W}_4 \cdot \mathbf{A}_4$ $\mathcal{O}(N_F^2 N^2 N_E N_G D)$
- The entries have to be re-ordered
  $\mathbf{J}[ik][jl] = \mathbf{K}[ij][kl]$ $\mathcal{O}(N_F^2 N^2 N_E)$

Benchmark

Evaluation of the Jacobian matrix for continuous Galerkin 2D diffusion
5746 triangles
MacBook Pro @2600MHz (single threaded)

p	t_{old}	t_{new}
1	0.06	0.08
2	0.39	0.13
3	1.76	0.25
4	5.96	0.39
5	16.51	0.61
6	42.51	0.95
7	96.47	1.43
8	196.49	2.33
9	373.01	4.28

Benchmark

Evaluation of the Jacobian matrix for continuous Galerkin 2D diffusion
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p	t_{old}	t_{new}	t_{solver}
1	0.06	0.08	0.06
2	0.39	0.13	0.41
3	1.76	0.25	0.93
4	5.96	0.39	1.60
5	16.51	0.61	2.39
6	42.51	0.95	3.78
7	96.47	1.43	5.13
8	196.49	2.33	7.45
9	373.01	4.28	10.13

Benchmark

Evaluation of the Jacobian matrix for continuous Galerkin 2D diffusion
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p	$N_G N^2$	t_{old}	t_{new}	t_{solver}
1	36	0.06	0.08	0.06
2	252	0.39	0.13	0.41
3	1,300	1.76	0.25	0.93
4	4,275	5.96	0.39	1.60
5	11,907	16.51	0.61	2.39
6	29,008	42.51	0.95	3.78
7	62,208	96.47	1.43	5.13
8	123,525	196.49	2.33	7.45
9	220,825	373.01	4.28	10.13

Benchmark

Evaluation of the Jacobian matrix for continuous Galerkin 2D diffusion
5746 triangles
MacBook Pro @2600MHz (single threaded)

p	$N_G N^2$	t_{old}	r_{old}	t_{new}	t_{solver}
1	36	0.06	167	0.08	0.06
2	252	0.39	155	0.13	0.41
3	1,300	1.76	135	0.25	0.93
4	4,275	5.96	140	0.39	1.60
5	11,907	16.51	139	0.61	2.39
6	29,008	42.51	147	0.95	3.78
7	62,208	96.47	156	1.43	5.13
8	123,525	196.49	160	2.33	7.45
9	220,825	373.01	169	4.28	10.13

$$r = \frac{t}{N_G N^2} 10^5$$

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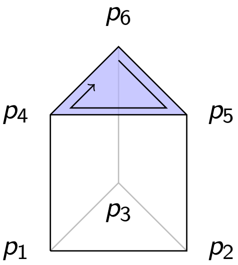
p	$N_G N^2$	t_{old}	r_{old}	t_{new}	r_{new}	t_{solver}
1	36	0.06	167	0.08	220	0.06
2	252	0.39	155	0.13	52	0.41
3	1,300	1.76	135	0.25	20	0.93
4	4,275	5.96	140	0.39	9.1	1.60
5	11,907	16.51	139	0.61	5.1	2.39
6	29,008	42.51	147	0.95	3.3	3.78
7	62,208	96.47	156	1.43	2.3	5.13
8	123,525	196.49	160	2.33	1.9	7.45
9	220,825	373.01	169	4.28	1.9	10.13

$$r = \frac{t}{N_G N^2} 10^5$$

Interface terms

Contributions from all faces with identical face-volume relative position are computed with one single BLAS product.



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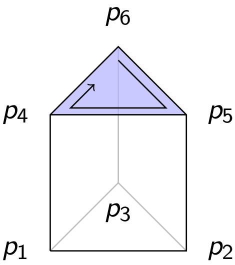
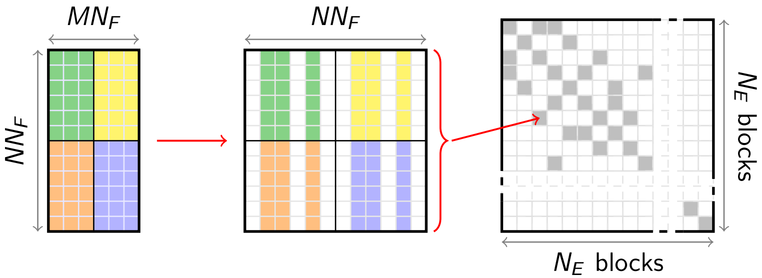


Matrix assembled by blocks



Discontinuous benchmark

Evaluate and assemble the Jacobian matrix
Discontinuous Galerkin 3D advection/diffusion
MacBook Pro @2600MHz (single threaded)

p	time/N_E	interface
1	$5.3\ 10^{-5}s$	71.7 %
2	$1.1\ 10^{-4}s$	62.5 %
3	$3.1\ 10^{-4}s$	51.1 %
4	$9.2\ 10^{-4}s$	40.4 %
5	$3.5\ 10^{-3}s$	26.6 %
6	$1.1\ 10^{-2}s$	19.9 %
7	$3.1\ 10^{-2}s$	15.2 %
8	$1.1\ 10^{-1}s$	9.4 %

Conclusion

- Recipe to efficiently compute high order FE integrals
- No compromize on integration precision
- 7 times faster for P3 triangles
- 90 times faster for P9 triangles