Novel LBM Scheme for Euler Equations

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Summary. Lattice Boltzmann Method (LBM) as a relatively new numerical method has recently achieved considerable success in simulating compressible fluid flows. A hybrid method, which is a combination of the usual Finite Volume Method (FVM) and the LBM, is proposed here to solve the Euler Equations. The inter-cell parameters are obtained by solving the local Riemann problem for each inter-cell using LBM. LBM is a non-iterative process and is thus numerically inexpensive as compared to the Godunov scheme. The proposed scheme is benchmarked for several 1-D problems such as the shock tube problem, the Roe test and the Riemann problem. The results obtained using this hybrid method are compared with those obtained using the Godunov scheme.

1 Introduction

Lattice Boltzmann Method (LBM) has developed as an alternative and promising numerical technique for simulating fluid flows, e.g., [1],[2],[3]. Boltzmann equation is solved to simulate the fluid flow using collision models such as the Bhatnagar-Gross-Krook (BGK) model [4]. The method is computationally less expensive and the boundary conditions can be easily implemented to get high accuracy [5]. It is very effective even with complex boundaries [6] and can also simulate steep shock and contact discontinuity profiles. One of the biggest disadvantages of LBM is that it cannot be used with a non-uniform mesh. FVM along with the use of the Godunov scheme to obtain the interface parameters has been widely used to simulate Euler equations [7]. The key ingredient of this scheme is the solution of a Riemann problem. There is no closed-form solution to the Riemann problem and iterative methods such as the Newton Raphson method are used to obtain the solution. FVM can be used with non-uniform mesh but the use of iterative schemes to obtain the solution of Riemann problem makes the method computationally expensive.

The proposed hybrid method in the present work solves the Euler equations using FVM with the solution to the Riemann problem being obtained by LBM. We employ the compressible perfect gas LB model proposed by Kataoka and Tsutahara [8]. This model allows for the free choice of specific heat ratio. LBM is a non-iterative solver for the Riemann problem and is therefore expected to be more efficient than Godunov scheme. Thus the hybrid method is expectecd to incorporate the advantages of both FVM and LBM.

2 Non-dimensional form of the Euler equations

All variables and equations are expressed in their non-dimensional form for the convenience of numerical calculations and analysis. Let L, ρ_{R0}, and T_{R0} respectively be the reference length, density and temperature. Then the other non-dimensional variables are defined as

$$t = \frac{\hat{t}}{L/\sqrt{RT_{R0}}},\ x = \frac{\hat{x}}{L},\ y = \frac{\hat{y}}{L} \tag{1}$$

$$\rho = \frac{\hat{\rho}}{\rho_{R0}},\ u_x = \frac{\hat{u_x}}{\sqrt{RT_{R0}}},\ u_y = \frac{\hat{u_y}}{\sqrt{RT_{R0}}},\ T = \frac{\hat{T}}{T_{R0}},\ p = \frac{\hat{p}}{\rho_{R0}RT_{R0}}. \tag{2}$$

The symbol ($\hat{}$) indicates a parameter in its dimensional form.

The two-dimensional time-dependent Euler equations in differential form are

$$U_t + F(U)_x + G(U)_y = 0, \tag{3}$$

where U, F(U) and G(U) are the vectors of conserved variables and fluxes given by

$$U = \begin{bmatrix} \rho \\ \rho u_x \\ \rho u_y \\ E \end{bmatrix},\ F(U) = \begin{bmatrix} \rho u_x \\ \rho u_x^2 + p \\ \rho u_x u_y \\ u_x(E+p) \end{bmatrix},\ G(U) = \begin{bmatrix} \rho u_y \\ \rho u_x u_y \\ \rho u_y^2 + p \\ u_y(E+p) \end{bmatrix}. \tag{4}$$

E (total internal energy per unit volume) is given by

$$E = \rho[0.5(u_x^2 + u_y^2) + e]. \tag{5}$$

Equation of state is given by

$$p = \rho T. \tag{6}$$

A conservative scheme for Eq. 3 is of the form

$$U_{(i,j)}^{n+1} = U_{(i,j)}^{n} + \frac{\Delta t}{\Delta x}[F_{i-1/2} - F_{i+1/2}] + \frac{\Delta t}{\Delta y}[G_{i-1/2} - G_{i+1/2}], \tag{7}$$

where $U_{(i,j)}^n$ denotes the parameters at the node (i,j) and at time interval n. The inter-cell numerical fluxes F and G are computed using the solution of local Riemann problem. Instead of solving the local Riemann problem in an iterative manner, the LBM scheme is used to calculate inter-cell parameters.

3 LBM scheme for compressible Euler equations

LBM proposed by Kataoka and Tsutahara is used to simulate the inter-cell parameters at the new time step from the known cell and inter-cell parameters at the earlier time step. The solution procedure is non-iterative and is expected to be more efficient than the Godunov scheme where the analytical solution of the local Riemann problem is obtained by iterative procedure.

We now present a synopsis of the LB model developed by Kataoka and Tsutahara. In the Lattice Boltzmann model, the macroscopic variables density, velocity and temperature are defined in terms of a particle velocity distribution function (f_i) as:

$$\begin{cases} \rho = \sum_{1 \le i \le I} f_i, \\ \rho u_x = \sum_{1 \le i \le I} f_i c_{xi}, \\ \rho u_y = \sum_{1 \le i \le I} f_i c_{yi}, \\ \rho[bT + u_x^2 + u_y^2] = \sum_{1 \le i \le I} f_i [c_{xi}^2 + c_{yi}^2 + \eta_i^2]. \end{cases} \tag{8}$$

Here, c_{xi} and c_{yi} are the molecular velocities of the particle moving in i^{th} direction (i = 1,2, ..., I = 9) and η_i is another variable which has been introduced to control the specific heat ratio. b is a constant which is expressed in terms of specific heat ratio (γ) as

$$b = \frac{2}{\gamma - 1}. \tag{9}$$

The kinetic equation of the non-dimensional form is

$$\frac{\partial f_i}{\partial t} + c_{xi}\frac{\partial f_i}{\partial x} + c_{yi}\frac{\partial f_i}{\partial y} = \frac{f_i^{eq}(\rho, u_x, u_y, T) - f_i}{\epsilon}, \tag{10}$$

where ϵ is the Knudsen number and the distribution function at the old time step is obtained from the equilibrium distribution function, i.e.,

$$f_i^0 = f_i^{eq}(\rho^0, u_x^0, u_y^0, T^0). \tag{11}$$

The two-dimensional Lattice Boltzmann Model is now presented:

$$(c_{xi}, c_{yi}) = \begin{cases} (0,0) & (i = 1) \\ v_1(\cos\frac{\pi i}{2}, \sin\frac{\pi i}{2}) & (i = 2,3,4,5) \\ v_2(\cos\frac{\pi(i+.5)}{2}, \sin\frac{\pi(i+.5)}{2}) & (i = 6,7,8,9) \end{cases} \tag{12}$$

and

$$\eta_i = \begin{cases} \eta_0 \ (i = 1) \\ 0 \quad (i = 2,3,...9) \end{cases} \tag{13}$$

where v_1, v_2 ($v_1 \neq v_2$) and η_0 are non-zero constants.

The local equilibrium velocity distribution function is defined as

$$f_i^{eq} = \rho(A_i + B_i(u_x c_{xi} + u_y c_{yi}) + D_i(u_x c_{xi} + u_y c_{yi})^2) \quad (i = 1,2,3,...9) \tag{14}$$

where

$$A_i = \begin{cases} \frac{b-2}{\eta_0^2}T & (i = 1) \\ \frac{1}{4(v_1^2 - v_2^2)}[-v_2^2 + ((b-2)\frac{v_2^2}{\eta_0^2} + 2)T + \frac{v_2^2}{v_1^2}(u_x^2 + u_y^2)] & (i = 2,3,4,5) \\ \frac{1}{4(v_2^2 - v_1^2)}[-v_1^2 + ((b-2)\frac{v_1^2}{\eta_0^2} + 2)T + \frac{v_1^2}{v_2^2}(u_x^2 + u_y^2)] & (i = 6,7,8,9) \end{cases} \tag{15}$$

$$B_i = \begin{cases} \frac{-v_2^2+(b+2)T+u_x^2+u_y^2}{2v_1^2(v_1^2-v_2^2)} \; (i=2,3,4,5) \\ \\ \frac{-v_1^2+(b+2)T+u_x^2+u_y^2}{2v_2^2(v_2^2-v_1^2)} \; (i=6,7,8,9) \end{cases} \tag{16}$$

$$D_i = \begin{cases} \frac{1}{2v_1^4} \; (i=2,3,4,5) \\ \\ \frac{1}{2v_2^4} \; (i=6,7,8,9) \end{cases} \tag{17}$$

The parameters in Eq. 12 and Eq. 13 are chosen to be $v_1 = 1$, $v_2 = 3$, $\eta_0 = 2$. Discretized form of Eq. 10 is used to determine the new inter-cell parameters from the old node and inter-cell parameters in a single time step. In the present work the discretization employed is first order forward in time and first order upwind in space.

4 Results

The hybrid method has been numerically benchmarked against several test cases. These include several one-dimensional initial value problems (IVP) in gas dynamics, demonstrating the generation and propagation of shock waves, rarefaction waves and contact discontinuity waves. Specific heat ratio of the gas in all cases is 1.4 (or $b = 5$) and the time step is set at $\Delta t = \epsilon/4$. All results from this hybrid scheme have been compared with the exact solutions and the results obtained from Godunov scheme.

4.1 The shock tube problem

The rupture of a diaphragm which separates two stationary gases generates a wave system that typically consists of a rarefaction wave, a contact discontinuity and a shock wave. The initial variables of shock tube problem are given by

$$(\rho_L, u_L, T_L) = (1, 0, 1) \quad (x < 0),$$
$$(\rho_R, u_R, T_R) = (5, 0, 1) \quad (x > 0).$$

The pressure and the density profiles within the shock tube at a non-dimensionalized time, $t = 1$, are shown in Fig. 1. It can be noted that the hybrid scheme is able to capture the wave structure more accurately as compared with the FVM with Godunov scheme.

4.2 The Riemann problem

The Riemann problem is a slight generalization of the shock tube problem in the sense that the two gases on either side of the diaphragm may not be stationary. The given initial conditions generates a centered wave function which consists of two shock waves. The initial variables of Riemann problem are given by

$$(\rho_L, u_L, T_L) = (1, 1, 1) \quad (x < 0),$$
$$(\rho_R, u_R, T_R) = (1, -1, 1) \quad (x > 0).$$

The pressure and the density profiles within the one-dimensional domain at a non-dimensionalized time, $t = 1$, are shown in Fig. 2. It can again be noted that the hybrid scheme is able to capture the wave structure more accurately as compared with the FVM with Godunov scheme.

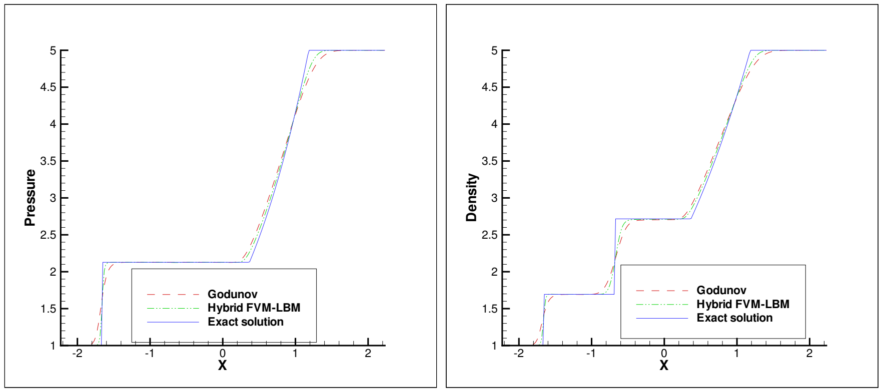

Fig. 1. Shock tube problem: pressure and density profile at $t = 1$ ($\Delta x = 0.02$; $\epsilon = 0.001$)

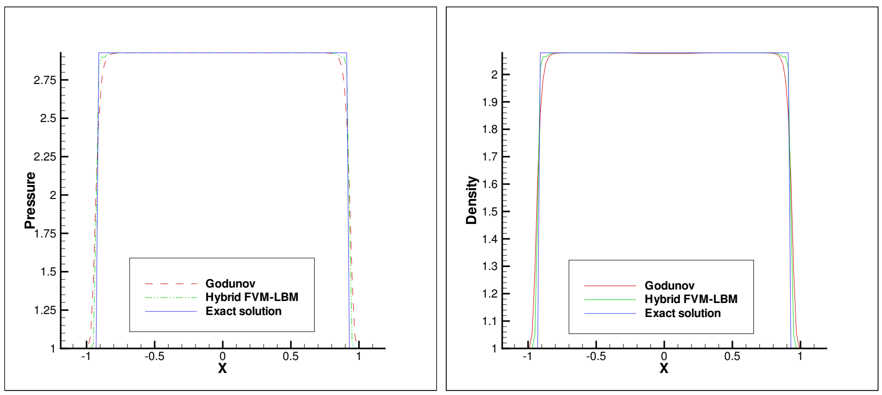

Fig. 2. Riemann problem: pressure and density profile at $t = 1$ ($\Delta x = 0.02$; $\epsilon = 0.001$)

4.3 The Roe test

The given initial condition generates a centered wave function which consists of rarefaction waves. The initial variables of Roe test are given by

$$(\rho_L, u_L, T_L) = (1, -1, 1.8) \qquad (x < 0),$$

$$(\rho_R, u_R, T_R) = (1, 1, 1.8) \qquad (x > 0).$$

The pressure and density profiles within the one-dimensional domain at a non-dimensionalized time, $t = 0.2$, are shown in Fig. 3. Once again it is seen that the hybrid scheme is able to capture the wave structure more accurately as compared with the FVM with Godunov scheme.

5 Conclusion

A hybrid method which is a combination of FVM and LBM is presented. The benchmarking tests show that the hybrid FVM-LBM scheme simulates the shock and contact discontinuity profile more accurately as compared to the profiles simulated by the

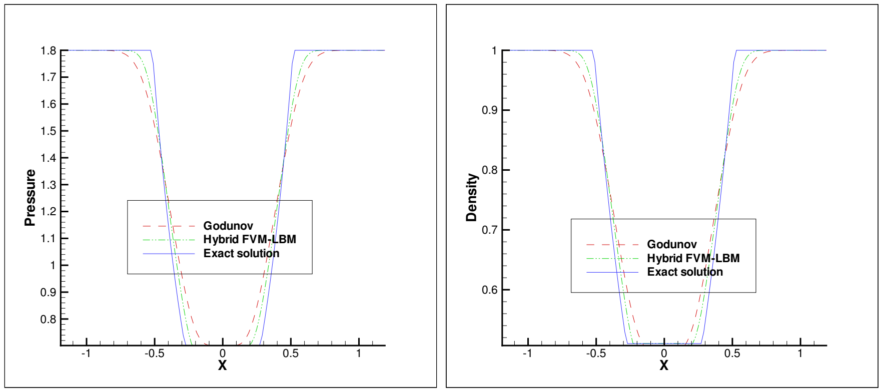

Fig. 3. Roe test: pressure and density profile at $t = 0.2$ ($\Delta x = 0.02$; $\epsilon = 0.001$)

Godunov scheme. The new scheme is computationally very efficient and this is clearly indicated by the reduced processing times as shown in the Table 1. In order to clearly determine the computational efficiency and accuracy, the hybrid scheme is required to be employed to simulate multidimensional problems; such efforts are underway.

Table 1. Comparison of processing times for the FVM-LBM scheme and the Godunov scheme

Benchmarking tests	FVM-LBM hybrid scheme	Godunov scheme
Shock tube problem	33.63 seconds	45.71 seconds
Riemann problem	34.78 seconds	46.44 seconds
Roe test (time = 0.2)	7.19 seconds	8.50 seconds

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