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2aAO5. The physical mechanism (viscosity related) of low frequency acoustic wave attenuation in sandy/silty sediments

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Considered saturated sediments contain diverse rock pebbles (characteristic size of 0.1 mm). The weight of higher pebbles holds lower pebbles in contact sufficiently that acoustically induced solid displacements vary slowly over several adjacent pebbles. Apart from contact areas, each is nearly surrounded by water at a nearly uniform pressure. An appropriate first approximation predicts that the elastic stress tensor in the pebbles is diagonal, with components equal to the negative of the acoustic pressure in the neighboring fluid. The assumptions of Mallock and Wood apply: the mass weighted local average velocity is proportional to the negative gradient of the pressure in the water. The no-slip condition at the interfaces tends to force the water to move with the pebbles, but the finite viscosity allows the fluid at small distances from the interfaces to move at a different velocity than the pebbles. The apparent driving force for the oscillations of the interstitial water relative to the pebbles is associated with the inertia of the water and is proportional to the difference in densities. The derived approximate wave equation predicts attenuation proportional to frequency squared, proportional to the square of the difference of the densities, and inversely proportional to viscosity.

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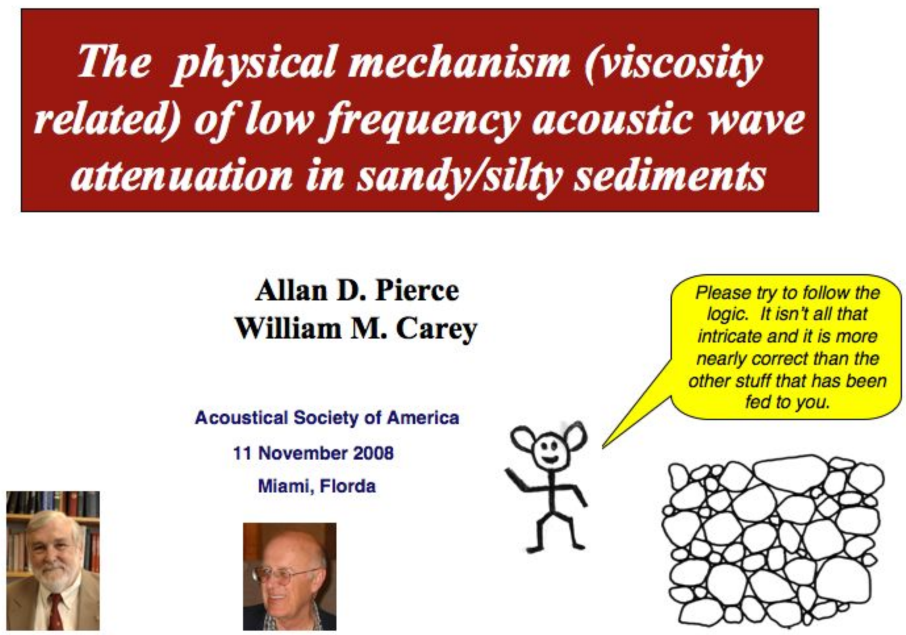
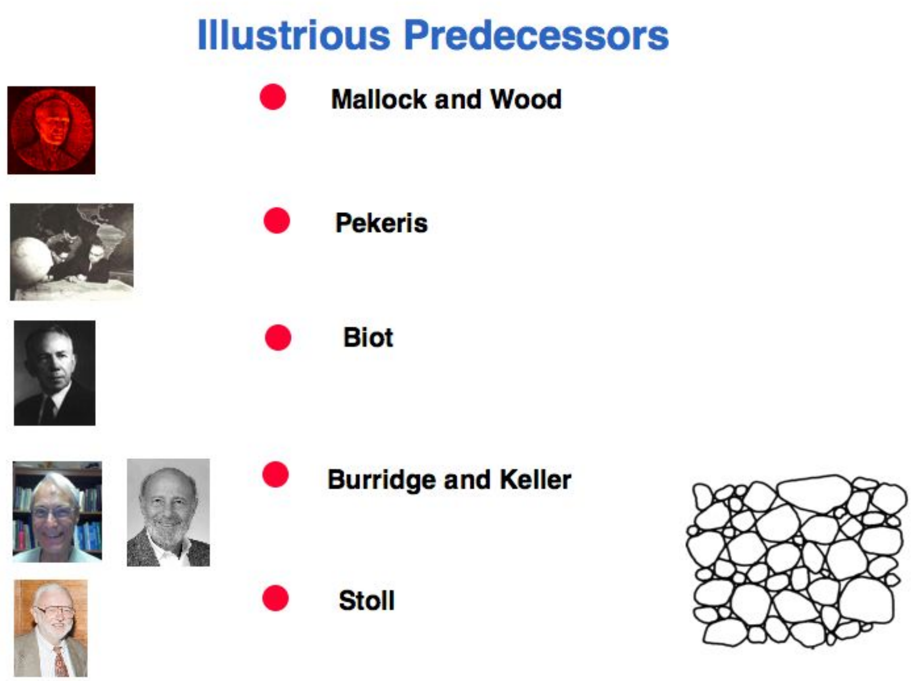
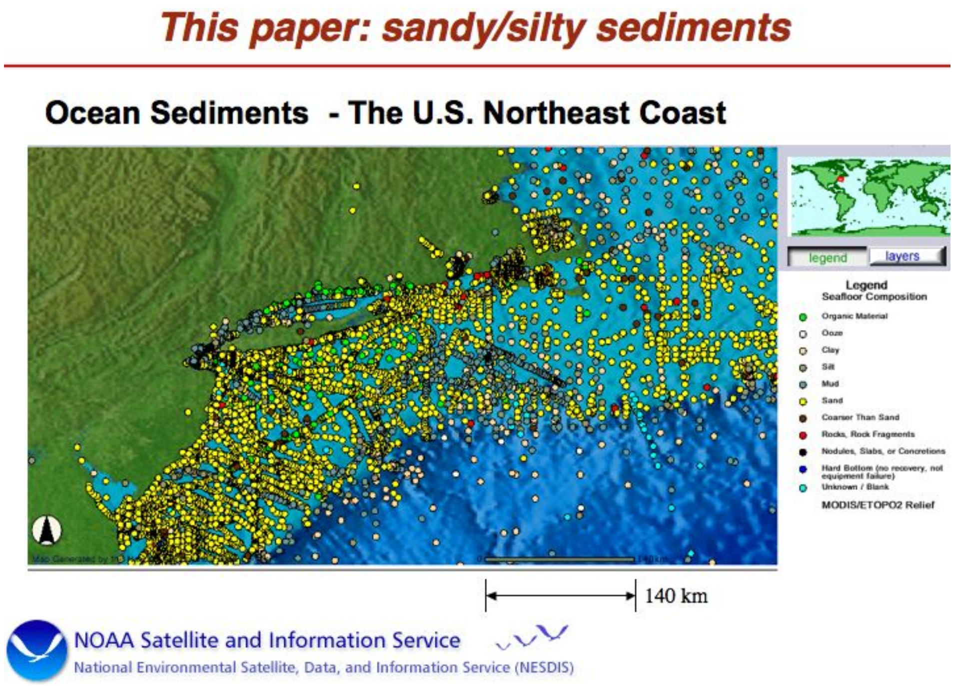
Portions of the present paper are similar to what appears in the paper: A. D. Pierce and W. M. Carey, *Low-frequency attenuation of acoustic waves in sandy/silty marine sediments*, J. Acoust. Soc. Amer. **124**, pp. EL-308–El-312 (November 2008). Some of the mathematical development here corrects and improves that of the JASA-EL paper.



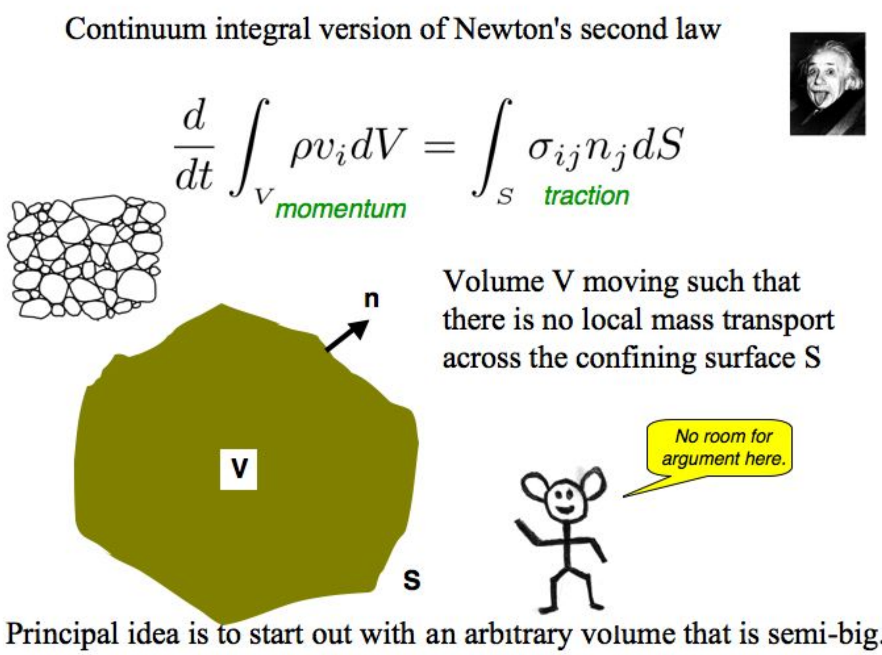
Detailed references can be found in the JASA-EL version.



In large portions of the world and especially off the northeast coast of the US, the sediments are of the sandy/silty type —- the present paper deals with this type of sediment.



In the mathematical expressions that appear here, the Einstein summation convention is use. The derivation begins with an integral version of Newton's second law.



Surface area in integral sufficiently large that

- **integrand can be replaced by local spatial averages**

$$\sigma_{ij} \rightarrow \sigma_{\mathrm{eff},ij} = \underbrace{\chi_f \langle \sigma_{ij} \rangle_f}_{\textit{fluid}} + \underbrace{\chi_s \langle \sigma_{ij} \rangle_s}_{\textit{solid}}$$

$\langle \quad \rangle_f$ = local average over portion of volume occupied by the fluid

$$\int_V \frac{\partial}{\partial t} \left[\chi_f \rho_f \langle v_i \rangle_f + \chi_s \rho_s \langle v_i \rangle_s \right] dV = \int_S \left[\chi_f \langle \sigma_{ij} \rangle_f + \chi_s \langle \sigma_{ij} \rangle_s \right] n_j dS$$

(Restriction to low-amplitude disturbances also imposed.)

"Rigorously derived" Cauchy equation of motion:

$$\rho_{\mathrm{eff}} \frac{\partial}{\partial t} v_{\mathrm{eff},i} = \frac{\partial}{\partial x_j} \sigma_{\mathrm{eff},ij}$$

(Spatial derivatives here are macro-derivatives.)

$$\sigma_{\mathrm{eff},ij} = \chi_f \langle \sigma_{ij} \rangle_f + \chi_s \langle \sigma_{ij} \rangle_s$$

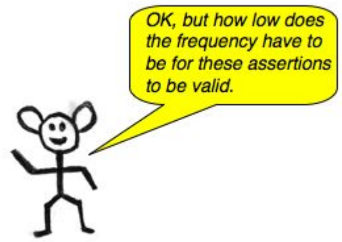
$$\rho_{\mathrm{eff}} = \chi_f \langle \rho \rangle_f + \chi_s \langle \rho \rangle_s$$

$$\rho_{\mathrm{eff}} \boldsymbol{v}_{\mathrm{eff}} = \chi_s \rho_s \langle \boldsymbol{v} \rangle_s + \chi_f \rho_f \langle \boldsymbol{v} \rangle_f$$

Approximations appropriate for low frequency acoustic disturbances

Hypothetical intermediate distance L:

- **L >> grain size**
- **acoustic wavelength >> L**



Over distances of order L:

- **motion is nearly uniform, both types of matter**
- **no-slip condition has strong effect**
- **displacements of fluid and solid portions nearly the same**

(Biot's "third type" of disturbance, where water diffuses through a solid frame-work, is ruled out by the latter restriction.)

Viscosity paradox

(at low frequencies)

Viscosity plus no-slip condition:

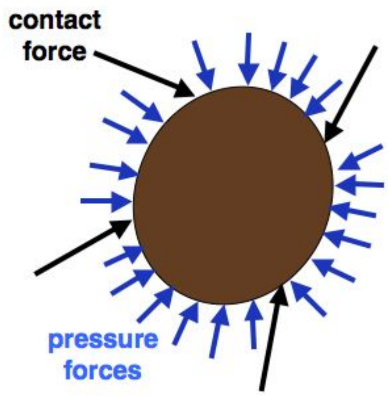
- **Causes fluid to move with the solid particles**
- **Causes solid particles to move with the fluid**
- **Causes shear gradients to be small**
- **Causes shear stress to be small in the fluid**

$$\langle \sigma_{ij} \rangle_f \approx -\langle p \rangle_f \delta_{ij}$$

approximation valid at sufficiently low frequencies

Focus on compressional wave disturbances

(shear effects mostly irrelevant)



Re: fluctuating part of stresses

$$\langle \sigma_{ij,\mathrm{shear}} \rangle_s \ll \langle \sigma_{ij,\mathrm{comp}} \rangle_s$$

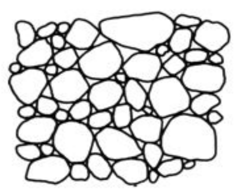
$$\langle \sigma_{ij} \rangle_s \approx -\langle p \rangle_f \delta_{ij}$$

Apparent Euler equation:

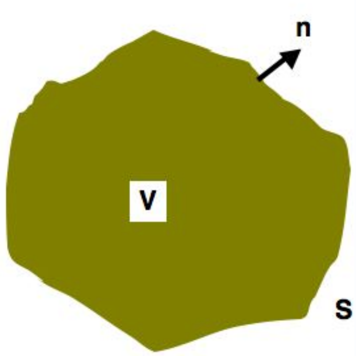
$$\rho_{\mathrm{eff}} \frac{\partial \boldsymbol{v}_{\mathrm{eff}}}{\partial t} = -\boldsymbol{\nabla} \langle p \rangle_f$$

Local bulk modulus B

$$\frac{d}{dt}\delta V = -\left[\frac{1}{B}\frac{\partial}{\partial t}\langle p \rangle_f\right]\delta V$$



(generalized Boyle-Mariotte law)



$$\frac{d}{dt}[\mathrm{Volume}] = -\int_V \left\{\frac{1}{B}\frac{\partial}{\partial t}\langle p \rangle_f\right\} dV = \int_S \boldsymbol{v} \cdot \boldsymbol{n}\, dS$$

Derivation of constitutive relation for sediment

$$-\int_V \left\{ \frac{1}{B} \frac{\partial}{\partial t} \langle p \rangle_f \right\} dV = \int_S \boldsymbol{v} \cdot \boldsymbol{n} \, dS$$

Local averaging of the integrands:

$$\frac{1}{B} \rightarrow \frac{\chi_s}{B_s} + \frac{\chi_f}{B_f} \qquad \boldsymbol{v} \rightarrow \chi_s \langle \boldsymbol{v} \rangle_s + \chi_f \langle \boldsymbol{v} \rangle_f$$

Application of the divergence theorem:

$$\int_S \left[\chi_s \langle \boldsymbol{v} \rangle_s + \chi_f \langle \boldsymbol{v} \rangle_f \right] \cdot \boldsymbol{n} \, dS = \int_V \boldsymbol{\nabla} \cdot \left[\chi_s \langle \boldsymbol{v} \rangle_s + \chi_f \langle \boldsymbol{v} \rangle_f \right] dV$$

Constitutive relation for sediment

$$\frac{\partial \langle p \rangle_f}{\partial t} = -B_{\text{eff}} \boldsymbol{\nabla} \cdot \left[\chi_s \langle \boldsymbol{v} \rangle_s + \chi_f \langle \boldsymbol{v} \rangle_f \right]$$

macro-divergence

$$B_{\text{eff}} = \left[\frac{\chi_s}{B_s} + \frac{\chi_f}{B_f} \right]^{-1}$$

effective bulk modulus

Effective Euler equation

$$\frac{\partial}{\partial t} \left[\chi_s \rho_s \langle \boldsymbol{v} \rangle_s + \chi_f \rho_f \langle \boldsymbol{v} \rangle_f \right] = -\boldsymbol{\nabla} \langle p \rangle_f$$

macro-gradient

Volume-weighted average velocity is not the same as mass-weighted average velocity. $\rho_s \neq \rho_f$

Change of symbols

$$\rho_{\text{eff}} \boldsymbol{v}_{\text{eff}} = \chi_s \rho_s \langle \boldsymbol{v} \rangle_s + \chi_f \rho_f \langle \boldsymbol{v} \rangle_f$$

$$\langle \boldsymbol{w} \rangle_f = \langle \boldsymbol{v} \rangle_f - \langle \boldsymbol{v} \rangle_s$$

$$\langle \boldsymbol{v} \rangle_f = \boldsymbol{v}_{\text{eff}} + \frac{\rho_s \chi_s}{\rho_{\text{eff}}} \langle \boldsymbol{w} \rangle_f \qquad \langle \boldsymbol{v} \rangle_s = \boldsymbol{v}_{\text{eff}} - \frac{\rho_f \chi_f}{\rho_{\text{eff}}} \langle \boldsymbol{w} \rangle_f$$

$$\frac{\partial}{\partial t} \langle p \rangle_f = -B_{\text{eff}} \boldsymbol{\nabla} \cdot \boldsymbol{v}_{\text{eff}} - \frac{B_{\text{eff}} \chi_s \chi_f}{\rho_{\text{eff}}} \left(\rho_s - \rho_f \right) \boldsymbol{\nabla} \cdot \langle \boldsymbol{w} \rangle_f$$

(small term)

$$\rho_{\text{eff}} \frac{\partial \boldsymbol{v}_{\text{eff}}}{\partial t} = -\boldsymbol{\nabla} \langle p \rangle_f$$

(Without the small term, you have standard field equations of acoustics --- Pekeris model)

Local fluid velocity in a "fluid region"

$$\rho_f \frac{\partial v_{f,i}}{\partial t} = -\frac{\partial}{\partial x_i} p + \eta \frac{\partial}{\partial x_j} \left[\frac{\partial v_{f,i}}{\partial x_j} + \frac{\partial v_{f,j}}{\partial x_i} - \frac{2}{3} \left(\boldsymbol{\nabla} \cdot \boldsymbol{v}_f \right) \delta_{ij} \right]$$

viscosity *rate of strain tensor*

Approximations consistent with low frequency limit:

On left side:

$$\boldsymbol{v}_f \rightarrow \langle \boldsymbol{v} \rangle_f = \boldsymbol{v}_{\text{eff}} + \frac{\rho_s \chi_s}{\rho_{\text{eff}}} \langle \boldsymbol{w} \rangle_f \qquad \frac{\partial \boldsymbol{v}_{\text{eff}}}{\partial t} \rightarrow -\frac{1}{\rho_{\text{eff}}} \boldsymbol{\nabla} \langle p \rangle_f$$

On right side:

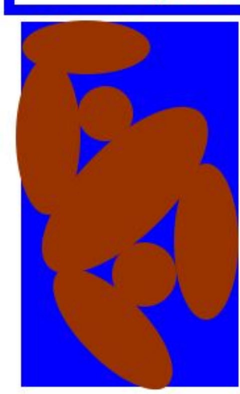
$$p \rightarrow \langle p \rangle_f \qquad \boldsymbol{v}_f \rightarrow \boldsymbol{w} + \langle \boldsymbol{v} \rangle_s$$

(negligible gradients)

After the approximations:

$$\frac{\chi_s(\rho_s-\rho_f)}{\rho_{\rm eff}}\boldsymbol{\nabla}\langle p\rangle_f+\frac{\rho_s\rho_f\chi_s}{\rho_{\rm eff}}\frac{\partial\langle\boldsymbol{w}\rangle_f}{\partial t}=\eta\left[\nabla^2\boldsymbol{w}+\frac{1}{3}\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{w}\right)\right]$$

(constant within domain) *(must satisfy no-slip condition)*



Formal solution:

$$w_i=-W_{ij}\frac{a^2}{\eta}\left[\frac{\chi_s(\rho_s-\rho_f)}{\rho_{\rm eff}}\frac{\partial\langle p\rangle_f}{\partial x_j}+\frac{\rho_s\rho_f\chi_s}{\rho_{\rm eff}}\frac{\partial\langle w_j\rangle_f}{\partial t}\right]$$

$$\nabla_\xi^2 W_{ij}+\frac{1}{3}\frac{\partial^2}{\partial\xi_i\partial\xi_k}W_{kj}=-\delta_{ij}$$

$$\xi_x=\frac{x-x_o}{a};\quad \xi_y=\frac{y-y_o}{a};\quad \xi_z=\frac{z-z_o}{a}$$

(technique similar to that used by Burridge and Keller)

Statistical isotropy:

$$\langle W_{ij}\rangle_f=\beta\delta_{ij}$$

(dimensionless parameter β is estimated later in the talk to be 0.0013)

Second constitutive relation:

$$\langle\boldsymbol{w}\rangle_f=-\left(\frac{\beta a^2\chi_s(\rho_s-\rho_f)}{\eta\rho_{\rm eff}}\right)\boldsymbol{\nabla}\langle p\rangle_f-\left(\frac{\beta a^2\chi_s\rho_s\rho_f}{\eta\rho_{\rm eff}}\right)\frac{\partial\langle\boldsymbol{w}\rangle_f}{\partial t}$$

(neglect at low frequencies)

Previously derived:

$$\frac{\partial}{\partial t}\langle p\rangle_f=-B_{\rm eff}\boldsymbol{\nabla}\cdot\boldsymbol{v}_{\rm eff}-\frac{B_{\rm eff}\chi_s\chi_f}{\rho_{\rm eff}}\left(\rho_s-\rho_f\right)\boldsymbol{\nabla}\cdot\langle\boldsymbol{w}\rangle_f$$

(small term)

$$\rho_{\rm eff}\frac{\partial\boldsymbol{v}_{\rm eff}}{\partial t}=-\boldsymbol{\nabla}\langle p\rangle_f$$

Estimate of upper frequency limit on model:

$$\langle \boldsymbol{w} \rangle_f = -\left(\frac{\beta a^2 \chi_s (\rho_s - \rho_f)}{\eta \rho_{\text{eff}}} \right) \boldsymbol{\nabla} \langle p \rangle_f - \left(\frac{\beta a^2 \chi_s \rho_s \rho_f}{\eta \rho_{\text{eff}}} \right) \frac{\partial \langle \boldsymbol{w} \rangle_f}{\partial t}$$

At what frequency is the last term comparable to 1/2-th of the term on the left?

$$f_{\text{upper}} \approx \frac{1}{4\pi} \left(\frac{\eta \rho_{\text{eff}}}{\beta a^2 \chi_s \rho_s \rho_f} \right)$$

(magnitude estimated later in the talk is 6.2 kHz)

Approximate low-frequency wave equation with attenuation:

$$\frac{\partial^2}{\partial t^2} \langle p \rangle_f - B_{\text{eff}} \boldsymbol{\nabla} \cdot \left\{ \frac{1}{\rho_{\text{eff}}} \boldsymbol{\nabla} \langle p \rangle_f \right\} = \frac{B_{\text{eff}}}{\eta} \left(\frac{\rho_s - \rho_f}{\rho_{\text{eff}}} \right)^2 \beta a^2 \chi_s^2 \chi_f \frac{\partial}{\partial t} \nabla^2 \langle p \rangle_f$$

Complex wave number:

$$k = \omega \left[\frac{B_{\text{eff}}}{\rho_{\text{eff}}} - i\omega \frac{B_{\text{eff}}}{\eta} \left(\frac{\rho_s - \rho_f}{\rho_{\text{eff}}} \right)^2 \beta a^2 \chi_s^2 \chi_f \right]^{-1/2} \qquad c_{\text{eff}} = \left(\frac{B_{\text{eff}}}{\rho_{\text{eff}}} \right)^{1/2}$$

Attenuation constant:

$$\alpha = K_\alpha \omega^2 \qquad K_\alpha = \frac{\beta a^2}{2 c_{\text{eff}}} \frac{\rho_{\text{eff}}}{\eta} \left(\frac{\rho_s - \rho_f}{\rho_{\text{eff}}} \right)^2 \chi_s^2 \chi_f$$

Relation of upper frequency limit to attenuation

$$f_{\text{upper}} \approx \frac{1}{4\pi}\left(\frac{\eta\rho_{\text{eff}}}{\beta a^2 \chi_s \rho_s \rho_f}\right)$$

$$K_\alpha = \frac{\beta a^2}{2c_{\text{eff}}}\frac{\rho_{\text{eff}}}{\eta}\left(\frac{\rho_s-\rho_f}{\rho_{\text{eff}}}\right)^2 \chi_s^2 \chi_f$$

$$\boxed{f_{\text{upper}} \approx \frac{\chi_s \chi_f}{8\pi c_{\text{eff}} K_\alpha}\frac{(\rho_s-\rho_f)^2}{\rho_s \rho_f}}$$

Nantucket Sound numbers

$$\frac{\eta}{\rho_f} = 10^{-6}\,\text{m}^2/\text{s} \qquad a = 0.115\,\text{mm} \qquad c_{\text{eff}} = 1600\,\text{m/s}$$

$$\chi_f = 0.49 \qquad \frac{\rho_s}{\rho_f} = 2.5 \qquad K_\alpha = \frac{0.015}{8.69[2\pi(220.5)]^2}\ \text{s}^2/\text{m}$$

Derived using theory in this paper:

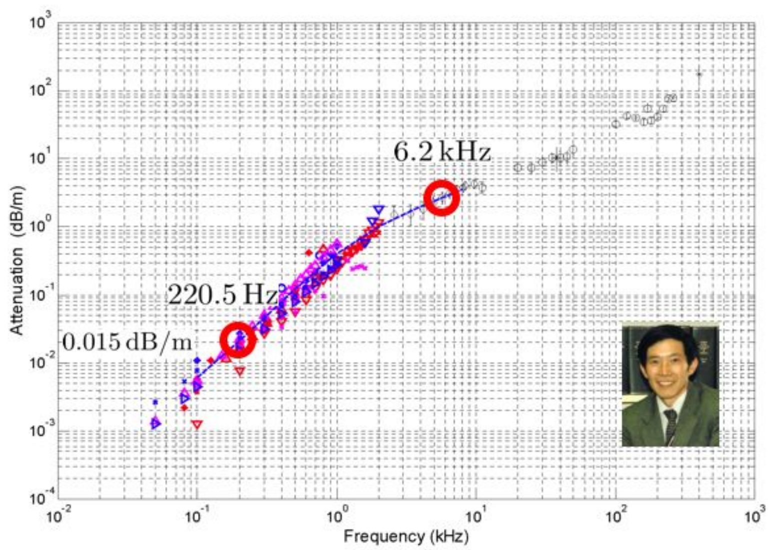
$$\beta = 0.0013 \qquad\qquad f_{\text{upper}} = 6.2\,\text{kHz}$$




The data shown in the graph below was assembled by Ji-Xun Zhou (Georgia Institute of Technology), who was the first speaker in the session in which the present paper was given. It is being used with Zhou's permission. The fact that the theory tends to predict where the break in the power law occurs allows a partial substantiation of the theory.

Composite experimental plot of attenuation versus frequency

- developed by Zhou



Concluding remarks

Attenuation constant:

$$\alpha = K_\alpha \omega^2 \qquad K_\alpha = \frac{\beta a^2}{2c_{\text{eff}}} \frac{\rho_{\text{eff}}}{\eta} \left(\frac{\rho_s - \rho_f}{\rho_{\text{eff}}} \right)^2 \chi_s^2 \chi_f$$

- **Proportionality to frequency-squared at low frequencies is very fundamental, easy to derive from causality considerations**
- **Inverse proportionality to viscosity was "sort of implicit" in Biot's original heuristic theory, but not widely appreciated.**
- **Proportionality to square of density difference is fundamental result of the viscous partial retardation mechanism. Prediction appears to not have been made before.**

(Explicit dependencies can by tested by laboratory scale experiments.)

(Parameter β can be estimated by sophisticated computation.)