2.29 Numerical Fluid Mechanics
Fall 2009 – Special Lecture 2

Discontinuous Galerkin Finite Element Methods

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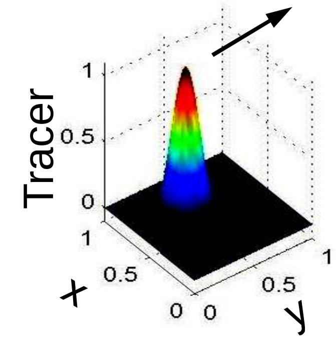
Presentation Outline

- Background/Motivation
  - Scalar Advection
  - Lock Exchange Problem
  - High-Order on unstructured meshes
- Definitions and Notation
- Method of Weighted Residuals – Comparison to FV
- Concept of Basis and Test Functions
  - Types of test functions
  - Types of basis functions
  - Continuous versus Discontinuous
- Worked example
- Difficulties and future research

Background/Motivation

- DG Advantages
  - Localized memory access
  - Higher order accuracy
  - Well-suited to adaptive strategies
  - Designed for advection dominated flows
  - Excellent for wave propagation
  - Can be used for complex geometries
- DG Disadvantages
  - Expensive?
  - Difficult to implement
  - Difficulty in treating higher-order derivatives

Why Higher Order?



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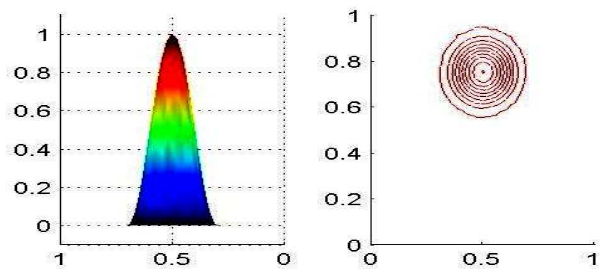
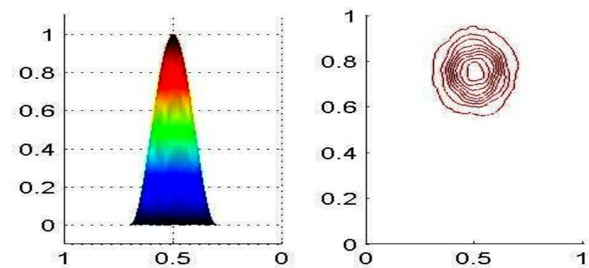
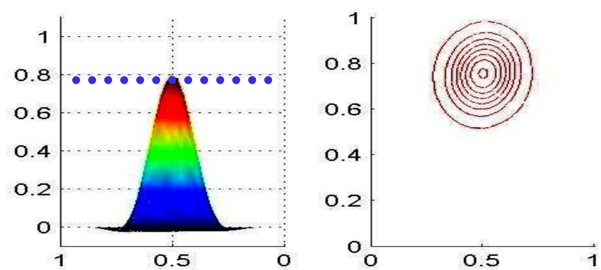
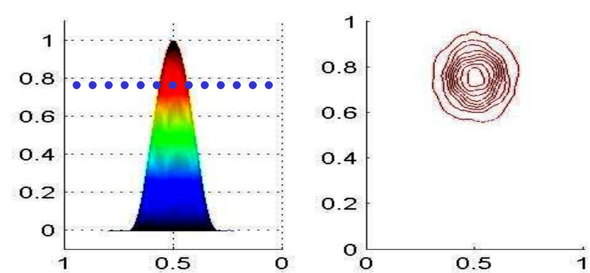
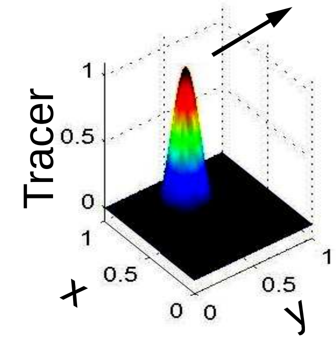
Initial Condition		
Final Condition		

Budgell W.P., et al.. Scalar advection schemes for ocean modelling on unstructured triangular grids. Ocean Dynamics (2007). Vol 57.

Why Higher Order?

- Less Numerical Diffusion/Dissipation
- Higher accuracy for lower computational time

- Example test case: 20 periods of linear tracer advection:



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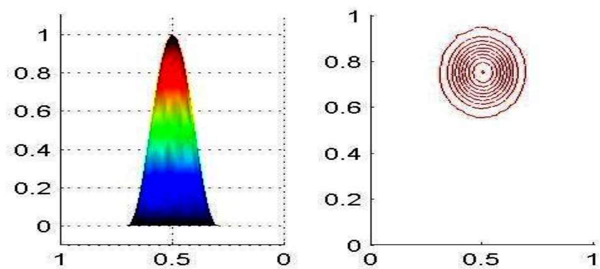
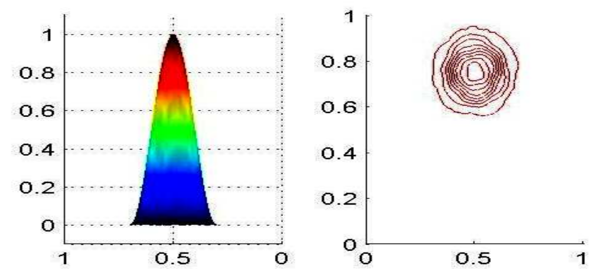
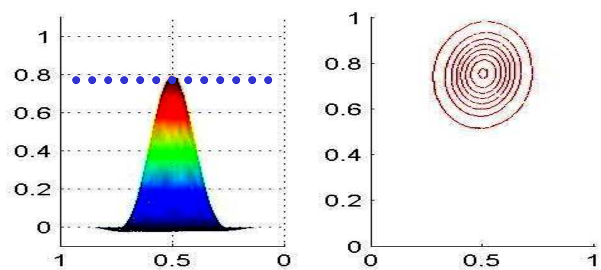
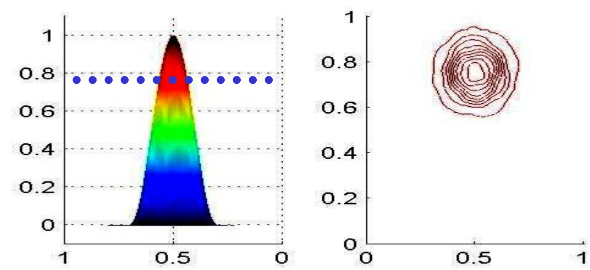
Initial Condition		
Final Condition		

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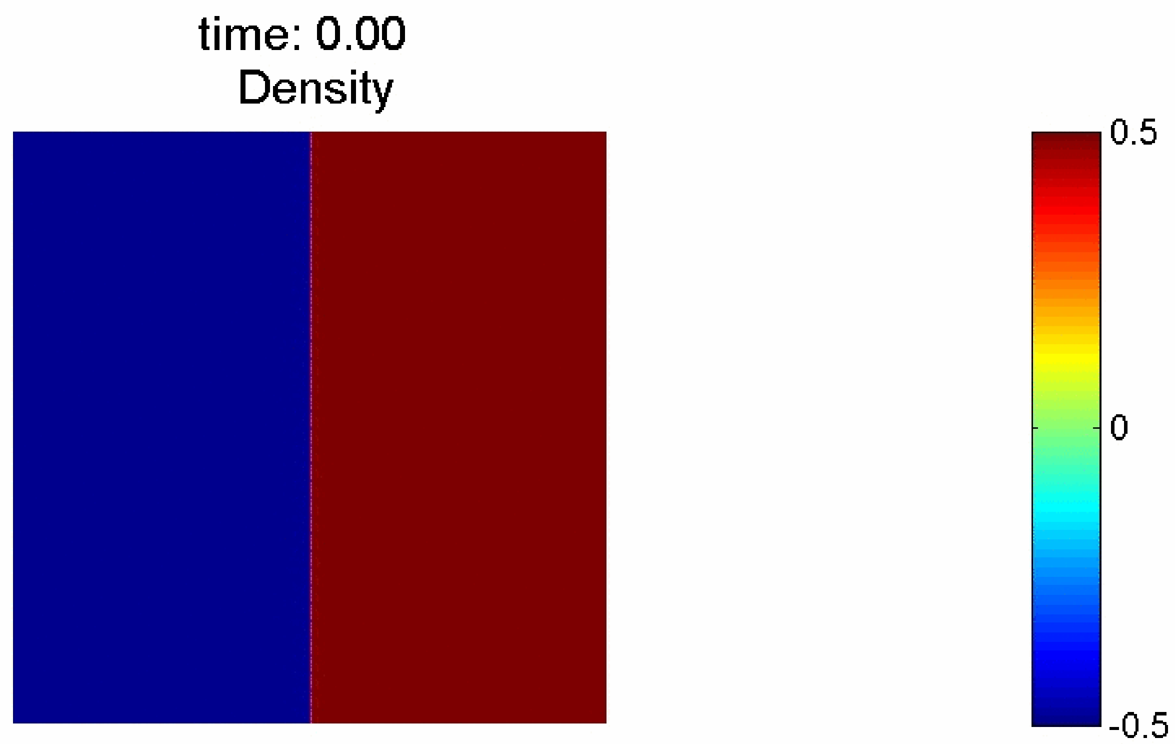
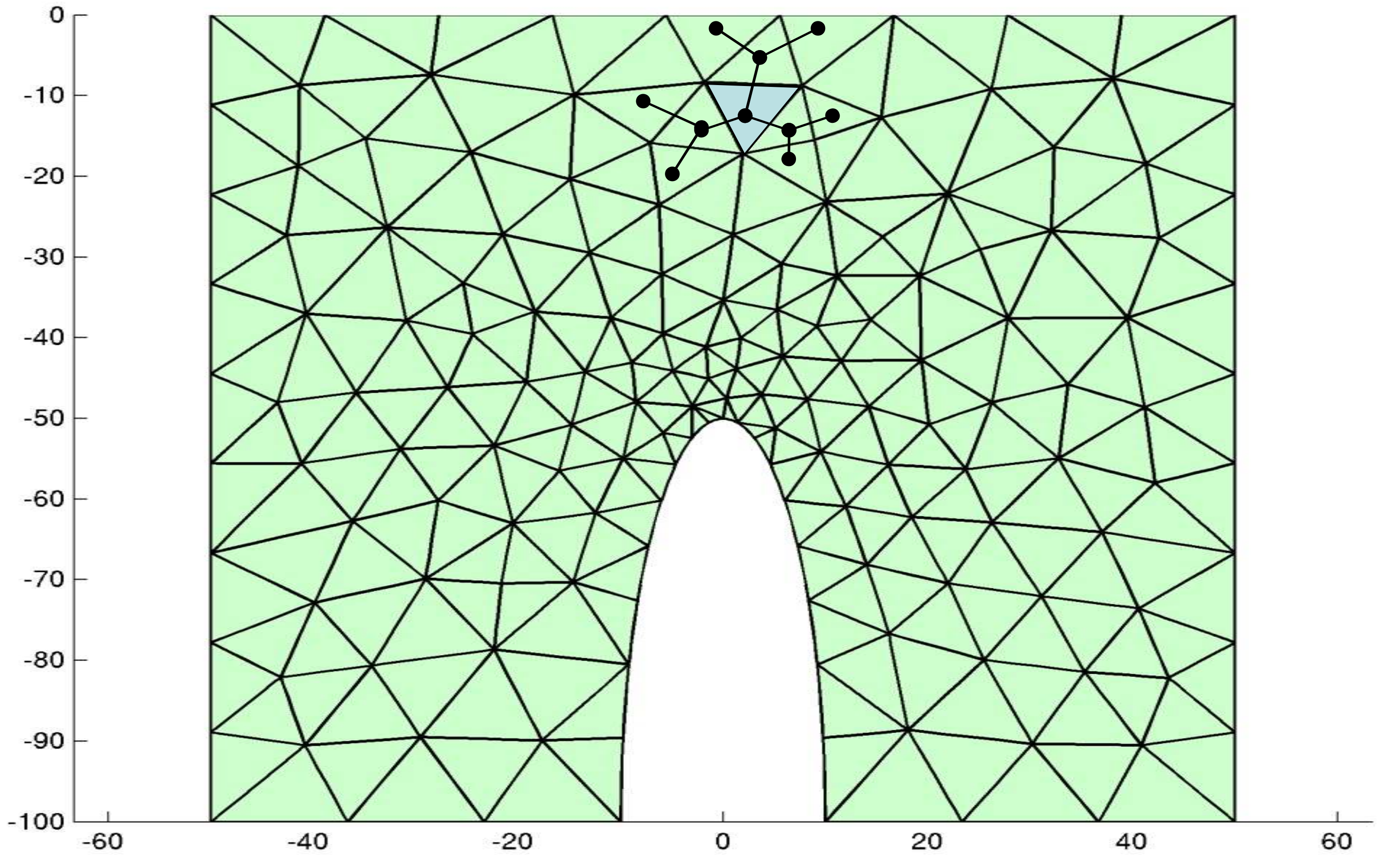
Higher Order Lock Exchange

- 5^{th} order elements
- 35 x 35 elements (equivalent to approx 230x230 FV)

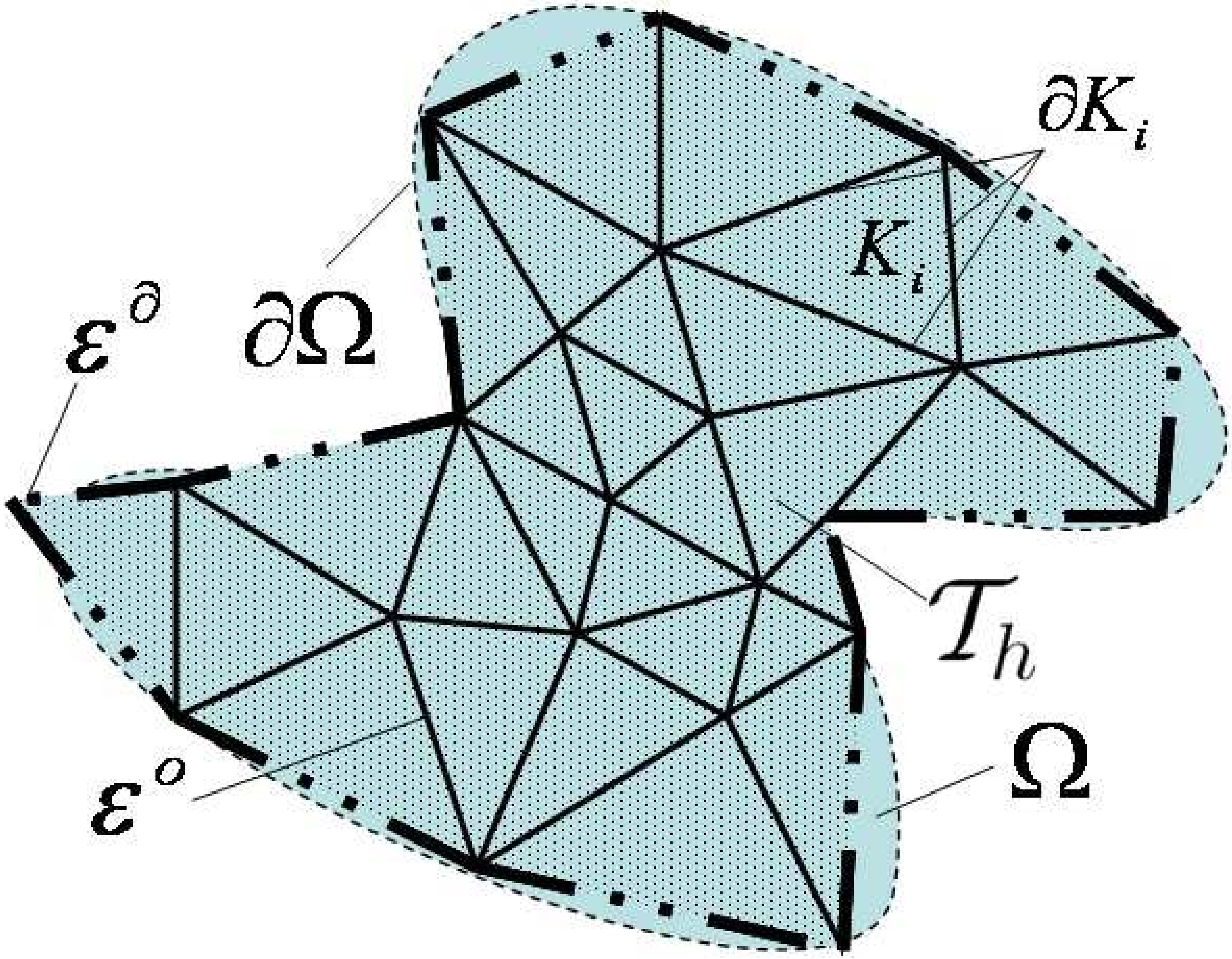


Higher Order on unstructured meshes

- Large Stencils are difficult
  - What to do at boundaries?



Definitions and Notation



Definitions and Notation

- General advection equation

$$\begin{aligned}
\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{F}(u) &= S(u), \quad \text{in } (0,T) \times \Omega \\
u &= g_D, \quad \text{on } \partial\Omega_D \\
\mathbf{F}(u) \cdot \hat{\mathbf{n}} &= g_N, \quad \text{on } \partial\Omega_N \\
u &= u_0, \quad \text{in } (0,0) \times \Omega
\end{aligned}$$

Method of Weighted Residuals (MWR)

- Multiply residual by test function
- Integrate over domain
- Set equal to zero

$$\int_K \left\{ \frac{\partial u_h}{\partial t} w + \left[\nabla \cdot \mathbf{F}(u_h)\right] w \right\} dK \;=\; \int_K S(u_h) w dK, \quad \forall K \in \mathcal{T}_h$$

- Integrate by parts

$$\int_K \frac{\partial u_h}{\partial t} w dK + \int_K \nabla \cdot \left[\mathbf{F}(u_h) w\right] dK - \int_K \mathbf{F}(u_h) \cdot \nabla w dK \;=\; \int_K S(u_h) w dK$$

- Divergence theorem (weak form)

$$\int_K \frac{\partial u_h}{\partial t} w dK + \int_{\partial K} \hat{\mathbf{F}}(u_h) \cdot \hat{\mathbf{n}} w d\partial K - \int_K \mathbf{F}(u_h) \cdot \nabla w dK \;=\; \int_K S(u_h) w dK$$

Test Functions-Function Spaces

- If test function is in infinite space
  - Exact minimization of residual
- Discretization of equations leads to finite test function space

$$W_h^p = \left\{ w \in L^2(\Omega) : w\,|_K \in \mathcal{P}^p(K), \forall K \in \mathcal{T}_h \right\}$$

w in L2 such that w restricted to K in polynomial space of order p, for all K in triangulation

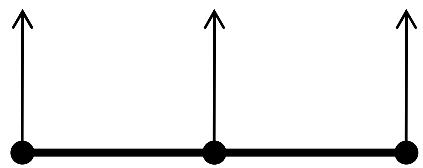
- Two spaces, the normed L^2 space and Hilbert space

- DG -- $L^2 : \quad \int_\Omega f(x)^2 d\Omega < \infty$
- CG -- $H^1 : \quad \int_\Omega f(x)^2 + \nabla f(x) \cdot \nabla f(x) d\Omega < \infty$

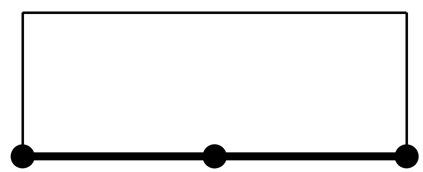
- Is x^2+x+1 in L^2? What about H^1? How about $\delta(x)$?

Test Functions

- Collocation



- Subdomain



- Galerkin
  - Test function is chosen to be the same as the basis function
  - Often used in practice

Basis Functions

- If basis in infinite space and test function in infinite space
  - Solution will be exact

$$u(\mathbf{x},t) \approx u_h(\mathbf{x},t) = \sum_i u_{h,i}(t)\theta_i(\mathbf{x})$$

$$u_h(\mathbf{x},t) = \sum_{i=1}^{N_P} u_{h,i}(t)\psi_i(\mathbf{x})$$

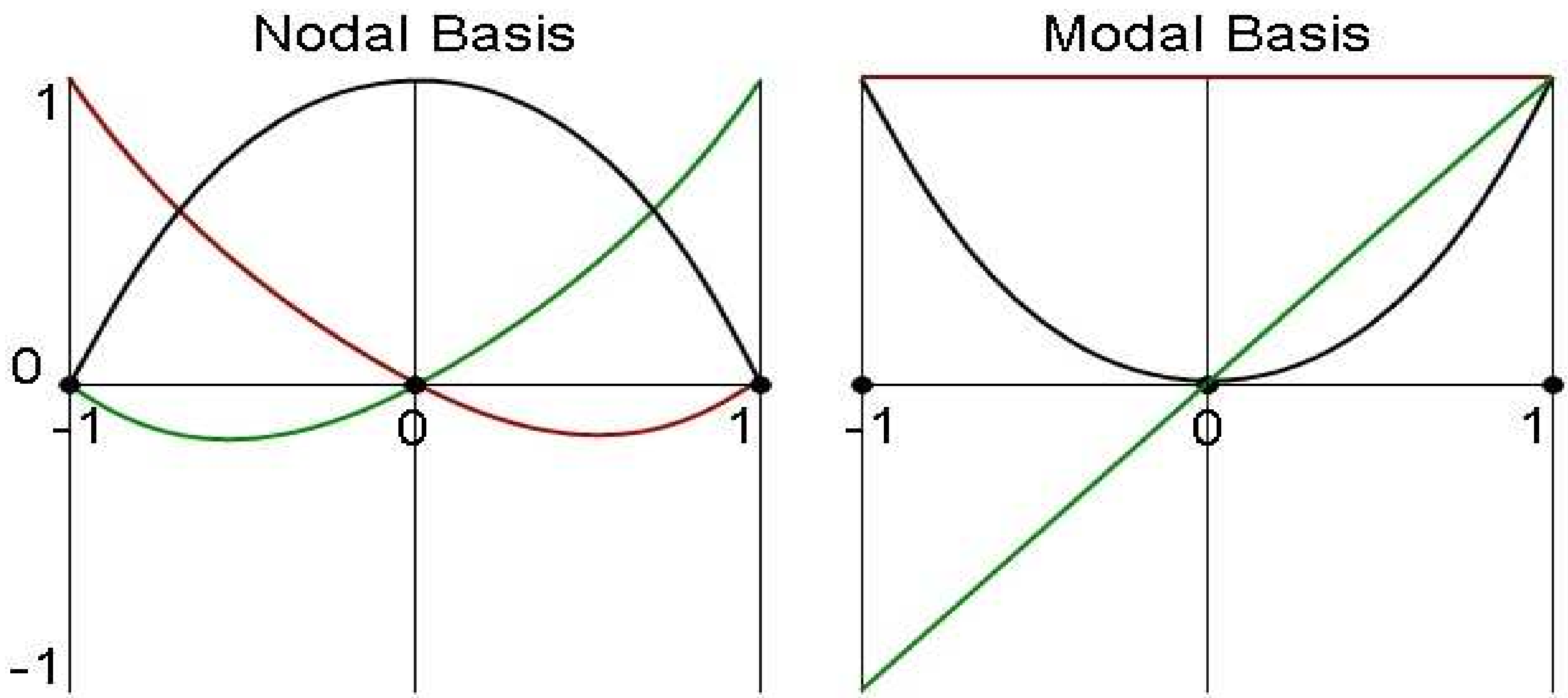
$$u_h(\mathbf{x},t) = \sum_{i=1}^{N_P} u_{h,i}(t,\mathbf{x}_i)\phi_i(\mathbf{x})$$

- Einstein Notation

$$u \approx u_i\theta_i$$

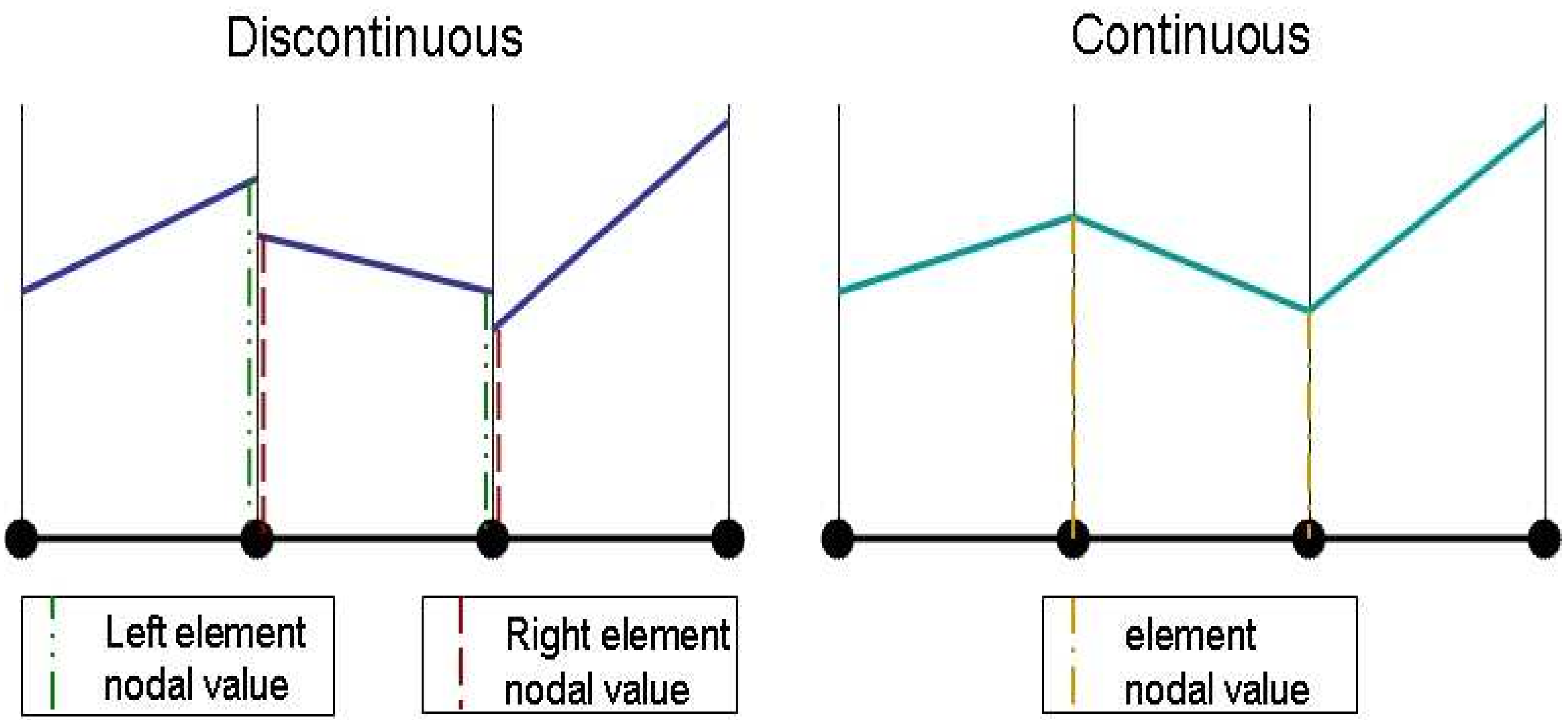
Basis Functions – Modal vs. Nodal

- Nodal
  1. $1/2X^2 - 1/2X$
  2. $1 - X^2$
  3. $1/2X^2 + 1/2x$
- Modal
  1. X^2
  2. X
  3. 1

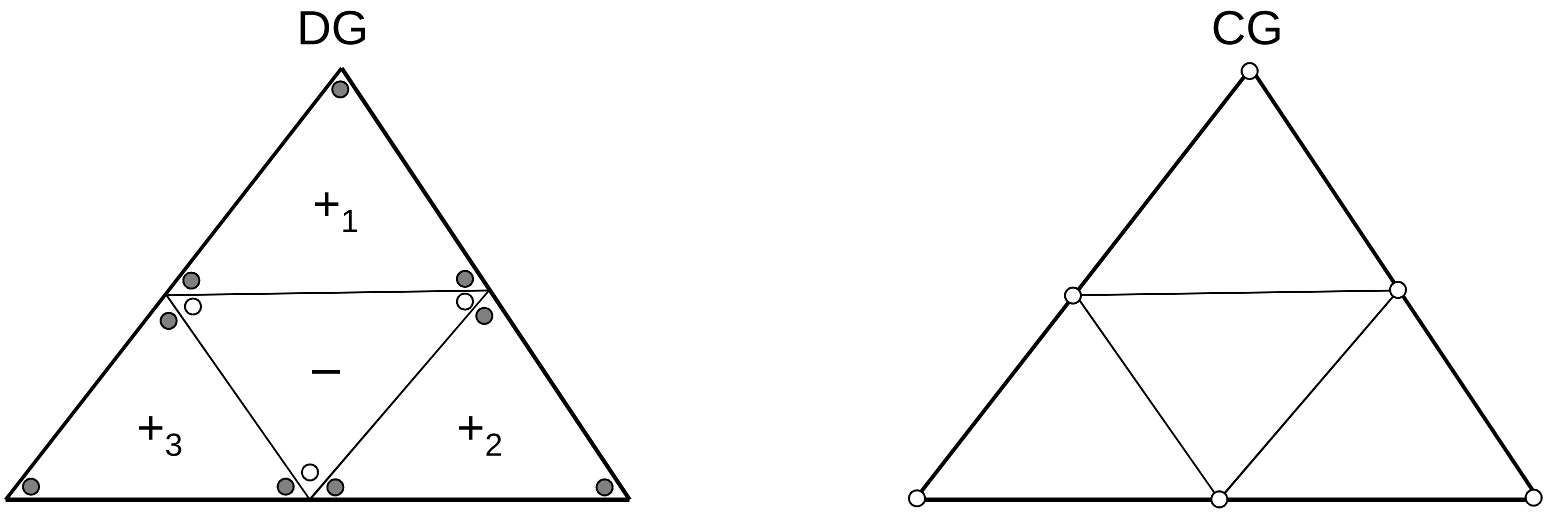


Basis Functions – Continuous vs. Discontinuous

- Continuous Function Space
- Discontinuous Function Space



Discontinuous vs Continuous Galerkin



- CG has continuity constraint at element edges
  - Forms matrix with many off-diagonal entries
  - Difficult to stabilize hyperbolic problems
- DG has no continuity constraint
  - Local solution in each element
  - Two unknowns on either side of element edges
  - Connection of domain achieved through fluxes: combination of unknowns on either side of edge
  - Forms matrix with block-diagonal structure

Worked Example

- Choose function space

$$W_h^p = \{w \in L^2(\Omega) : w|_K \in \mathcal{P}^p(K), \forall K \in \mathcal{T}_h\}$$

- Apply MWR

$$\begin{aligned}
\frac{\partial u}{\partial t} + \nabla \cdot (\vec{c}u) &= 0 \\
\int_K w\frac{\partial u}{\partial t} dK + \int_K w\nabla \cdot (\vec{c}u) dK &= 0 \\
\int_K w\frac{\partial u}{\partial t} dK - \int_K \nabla w \cdot (\vec{c}u) dK + \int_K \nabla \cdot (\vec{c}uw) dK &= 0 \\
\int_K w\frac{\partial u}{\partial t} dK - \int_K \nabla w \cdot (\vec{c}u) dK + \int_{\partial K} w\hat{n} \cdot \vec{c}\hat{u} d\partial K &= 0
\end{aligned}$$

Worked Example

- Substitute in basis and test functions

$$\int_K w\frac{\partial u}{\partial t}dK - \int_K \nabla w \cdot (\vec{c}u)dK + \int_{\partial K} w\hat{n}\cdot\vec{c}\hat{u}d\partial K = 0$$

$$\int_K w\frac{\partial u_j}{\partial t}\theta_j dK - \int_K \nabla w \cdot (\vec{c}u_j\theta_j)dK + \int_{\partial K} w\hat{n}\cdot(\vec{c}\hat{u}_j\theta_j)d\partial K = 0$$

$$\int_K \theta_i\frac{\partial u_j}{\partial t}\theta_j dK - \int_K \nabla \theta_i \cdot (\vec{c}u_j\theta_j)dK + \int_{\partial K} \theta_i n\cdot(\vec{c}\hat{u}_j\theta_j)d\partial K = 0$$

$$\frac{\partial u_j}{\partial t}\int_K \theta_i\theta_j dK - \vec{c}u_j\int_K \nabla \theta_i \cdot (\theta_j)dK + \vec{c}\hat{u}_j\cdot\int_{\partial K} \hat{n}(\theta_j\theta_i)d\partial K = 0$$

Worked Example

- Substitute for matrices
  - **M**- Mass matrix
  - **K**- Stiffness matrix or Convection matrix
- Solve specific case of 1D equations

$$\frac{\partial u_j}{\partial t}\int_K \theta_i\theta_j dK - \vec{c}u_j\int_K \nabla\theta_i\cdot(\theta_j)dK + \vec{c}\hat{u}_j\cdot\int_{\partial K}\hat{n}(\theta_j\theta_i)d\partial K = 0$$

$$\mathrm{M}_{ij}\frac{\partial u_j}{\partial t} - \mathbf{K}_{ij}\vec{c}u_j + \vec{c}\hat{u}_j\cdot\int_{\partial K}\hat{n}(\theta_j\theta_i)d\partial K = 0$$

$$1D: \quad \mathrm{M}_{ij}\frac{\partial u_j}{\partial t} - \mathbf{K}_{ij}\vec{c}u_j + \vec{c}\hat{u}_j\cdot\hat{n}(\theta_j\theta_i) = 0$$

$$1D: \quad u_j^{n+1} = u_j^n + \Delta t\mathrm{M}^{-1}(\mathbf{K}_{ij}\vec{c}u_j - \vec{c}\hat{u}_j\cdot\hat{n}\delta_{ij})$$

$$\hat{u}_j = \frac{u_j^+ + u_h^-}{2} - \frac{c}{|c|}\frac{u_j^+ - u_h^-}{2}$$

Worked Example

$$\frac{\partial u}{\partial t} + \nabla \cdot (\vec{c}u) = 0 \tag{1.1}$$

$$\int_K w\frac{\partial u}{\partial t}dK + \int_K w\nabla \cdot (\vec{c}u)dK = 0 \tag{1.2}$$

$$\int_K w\frac{\partial u}{\partial t}dK - \int_K \nabla w \cdot (\vec{c}u)dK + \int_K \nabla \cdot (\vec{c}uw)dK = 0 \tag{1.3}$$

$$\int_K w\frac{\partial u}{\partial t}dK - \int_K \nabla w \cdot (\vec{c}u)dK + \int_{\partial K} w\hat{n} \cdot \vec{c}\hat{u}d\partial K = 0 \tag{1.4}$$

$$\int_K w\frac{\partial u_j}{\partial t}\theta_j dK - \int_K \nabla w \cdot (\vec{c}u_j\theta_j)dK + \int_{\partial K} w\hat{n} \cdot (\vec{c}\hat{u}_j\theta_j)d\partial K = 0 \tag{1.5}$$

$$\int_K \theta_i\frac{\partial u_j}{\partial t}\theta_j dK - \int_K \nabla\theta_i \cdot (\vec{c}u_j\theta_j)dK + \int_{\partial K} \theta_i n \cdot (\vec{c}\hat{u}_j\theta_j)d\partial K = 0 \tag{1.6}$$

$$\frac{\partial u_j}{\partial t}\int_K \theta_i\theta_j dK - \vec{c}u_j\int_K \nabla\theta_i \cdot (\theta_j)dK + \vec{c}\hat{u}_j \cdot \int_{\partial K} \hat{n}(\theta_j\theta_i)d\partial K = 0 \tag{1.7}$$

$$\mathbf{M}_{ij}\frac{\partial u_j}{\partial t} - \mathbf{K}_{ij}\vec{c}u_j + \vec{c}\hat{u}_j \cdot \int_{\partial K} \hat{n}(\theta_j\theta_i)d\partial K = 0 \tag{1.8}$$

$$1D: \quad \mathbf{M}_{ij}\frac{\partial u_j}{\partial t} - \mathbf{K}_{ij}\vec{c}u_j + \vec{c}\hat{u}_j \cdot \hat{n}(\theta_j\theta_i) = 0 \tag{1.9}$$

$$1D: \quad u_j^{n+1} = u_j^n + \Delta t\mathrm{M}^{-1}(\mathbf{K}_{ij}\vec{c}u_j - \vec{c}\hat{u}_j \cdot \hat{n}\delta_{ij}) \tag{1.10}$$

$$\hat{u}_j = \frac{u_j^+ + u_h^-}{2} - \frac{c}{|c|}\frac{u_j^+ - u_h^-}{2} \tag{1.11}$$

Worked Example

```
clear all, clc, clf, close all

syms x
%create nodal basis
%Set order of basis function
%N >=2
N = 3;

%Create basis
if N==3
 theta = [1/2*x^2-1/2*x;
            1- x^2;
         1/2*x^2+1/2*x];
else
xi = linspace(-1,1,N);
 for i=1:N
  theta(i)=sym('1');
  for j=1:N
   if j~=i
    theta(i) = ...
     theta(i)*(x-xi(j))/(xi(i)-xi(j));
   end
  end
 end
end
```

```
%Create mass matrix
for i = 1:N
 for j = 1:N
  %Create integrand
  intgr  = int(theta(i)*theta(j));
  %Integrate
  M(i,j) =...
          subs(intgr,1)-subs(intgr,-1);
 end
end
%create convection matrix
for i = 1:N
 for j = 1:N
  %Create integrand
  intgr = ...
          int(diff(theta(i))*theta(j));
  %Integrate
  K(i,j) = ...
          subs(intgr,1)-subs(intgr,-1);
 end
end
```

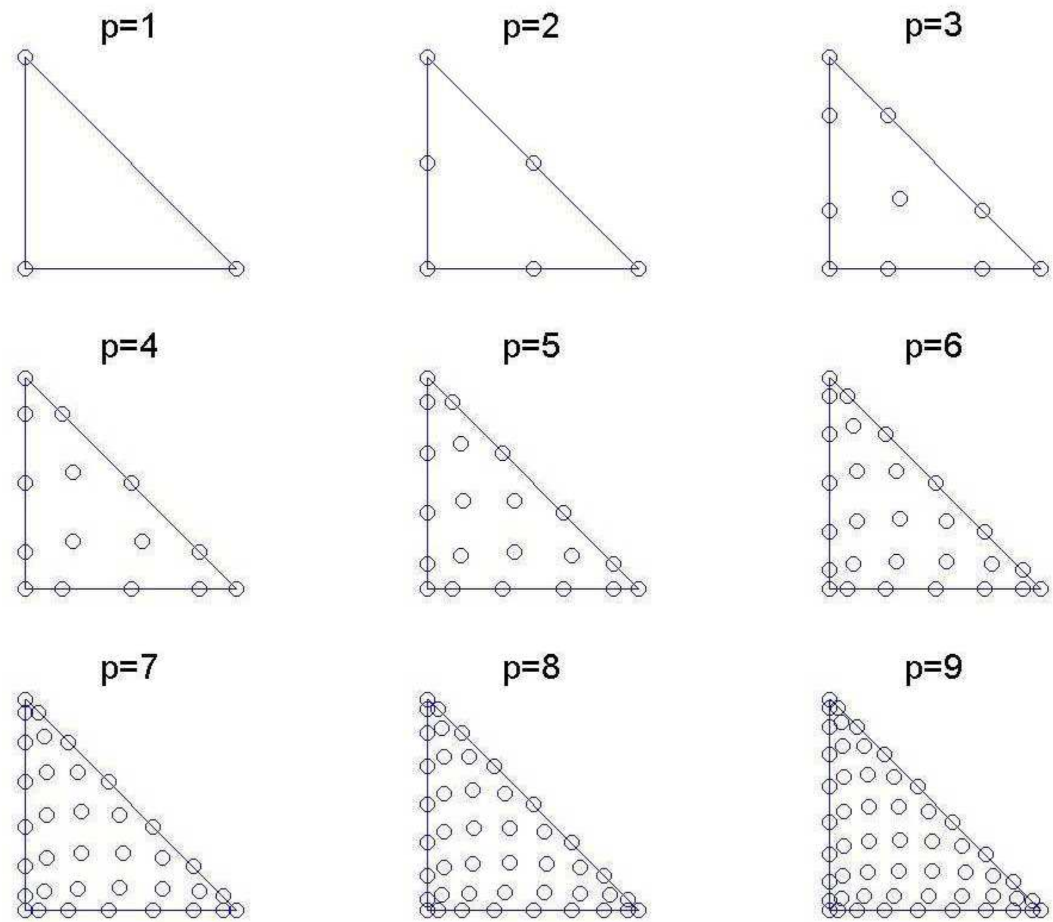
Worked Example

```
%% Initialize u
Nx = 20;
dx = 1./Nx;
%Multiply Jacobian through mass matrix.
%Note computationl domain has length=2,
actual domain length = dx
M=M*dx/2;

%Create "mesh"
x = zeros(N,Nx);
for i = 1:N
    x(i,:) =...
 dx/(N-1)*(i-1):dx:1-dx/(N-1)*(N-i);
end
%Initialize u vector
u = exp(-(x-.5).^2/.1^2);

%Set timestep and velocity
dt=0.002;   c=1;
%Periodic domain
ids = [Nx,1:Nx-1];
```

```
%Integrate over time
for i = 1:10/dt
    u0=u;
    %Integrate with 4th order RK
  for irk=4:-1:1
    %Always use upwind flux
    r = c*K*u;
    %upwinding
    r(end,:) = r(end,:) - c*u(end,:);
    %upwinding
    r(1,:) = r(1,:) + c*u(end,ids);
    %RK scheme
    u = u0 + dt/irk*(M\r);
  end
  %Plot solution
  if ~mod(i,10)
        plot(x,u,'b')
        drawnow
  end
end
```

Difficulties

- How to create basis on triangles, tetrahedrals?
  - Need to create set of well-behaved Nodal point
- Integration in 2D, 3D?
  - Higher-order quadrature rules on triangles, tetrahedrals



Difficulties and future research

- Higher-order derivatives
  - Naturally handled with CG
  - Somewhat more difficult with DG

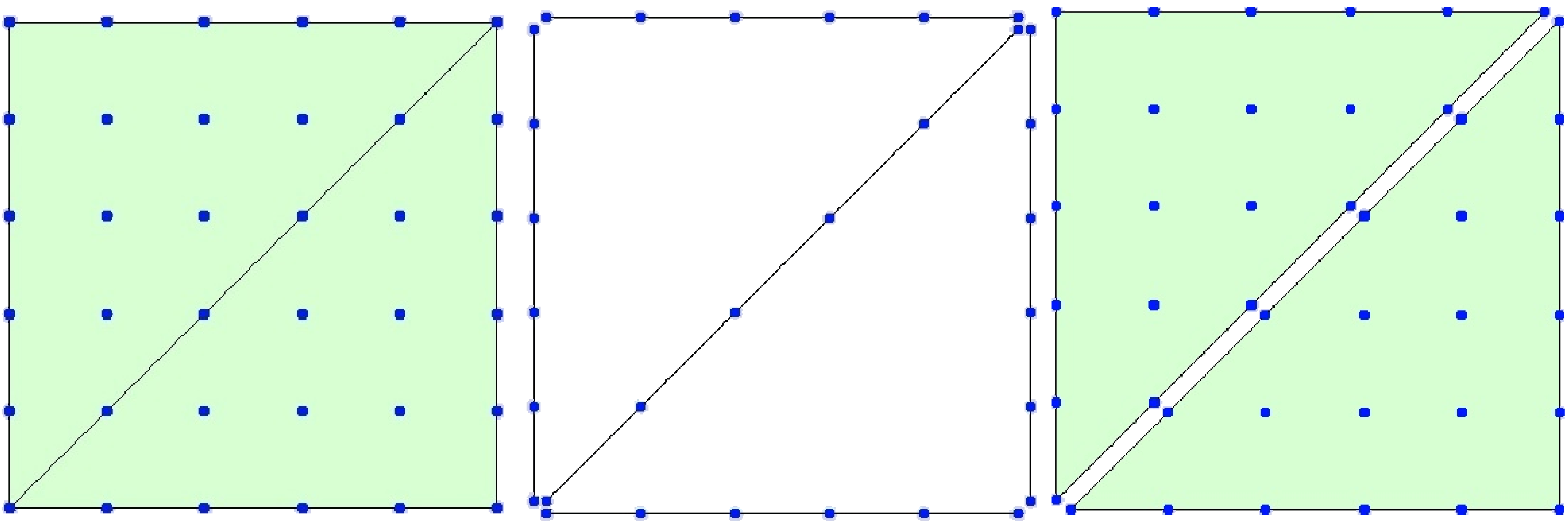
$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{F}_{inv}(u) + \nabla \cdot \mathbf{F}_{vis}(u,\ \nabla u) \;=\; S(u)$$

- Decompose higher derivatives into system of first-order derivatives

$$\begin{aligned}
\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{q} &= 0, \quad \text{in } (0,T) \times \Omega \\
\mathbf{q} + \kappa \nabla u &= 0, \quad \text{in } \Omega \\
u &= g_D, \quad \text{on } \partial\Omega_D \\
\mathbf{q} \cdot \hat{\mathbf{n}} &= g_N, \quad \text{on } \partial\Omega_N \\
u &= u_0, \quad \text{in } (0,0) \times \Omega
\end{aligned}$$

Difficulties and future research

- Research directed towards improved treatment of higher-order derivatives



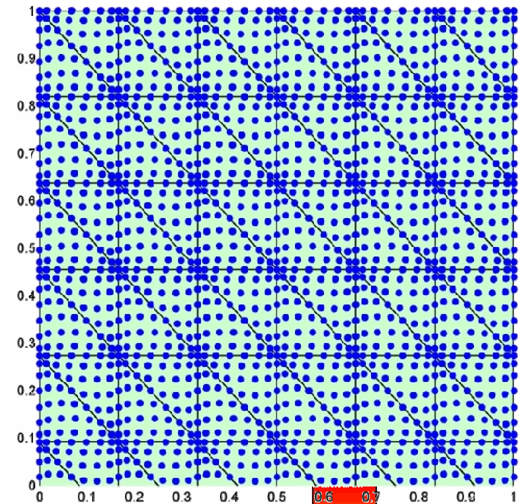
Continuous Galerkin — No duplication of DOF

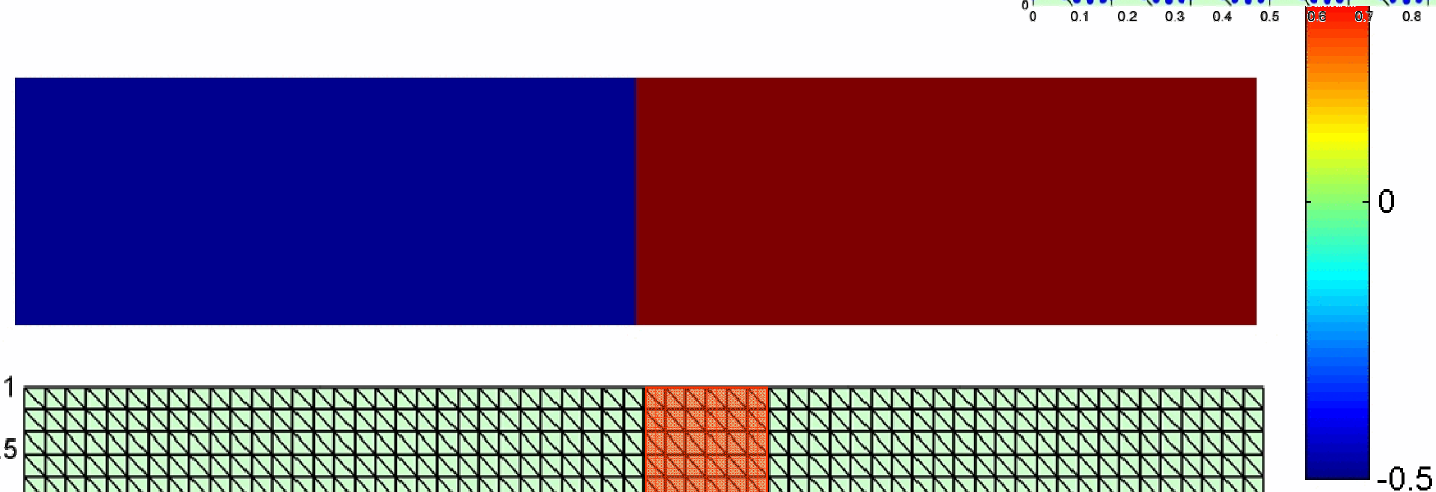
Hybrid Discontinuous Galerkin — Duplication at corners, but no interior DOFs!

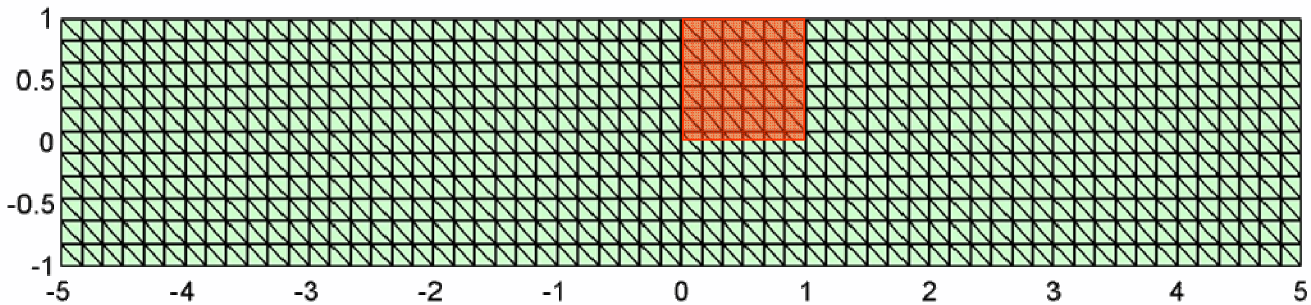
Discontinuous Galerkin — Duplication at edges

Lock Exchange Problem using HDG

- 37,000 DOF, 14,000 HDG unknowns
- 13.5 hrs
- 1320 Elements
- *p*=6
- Gr = 1.25×10^6, Sc=0.71

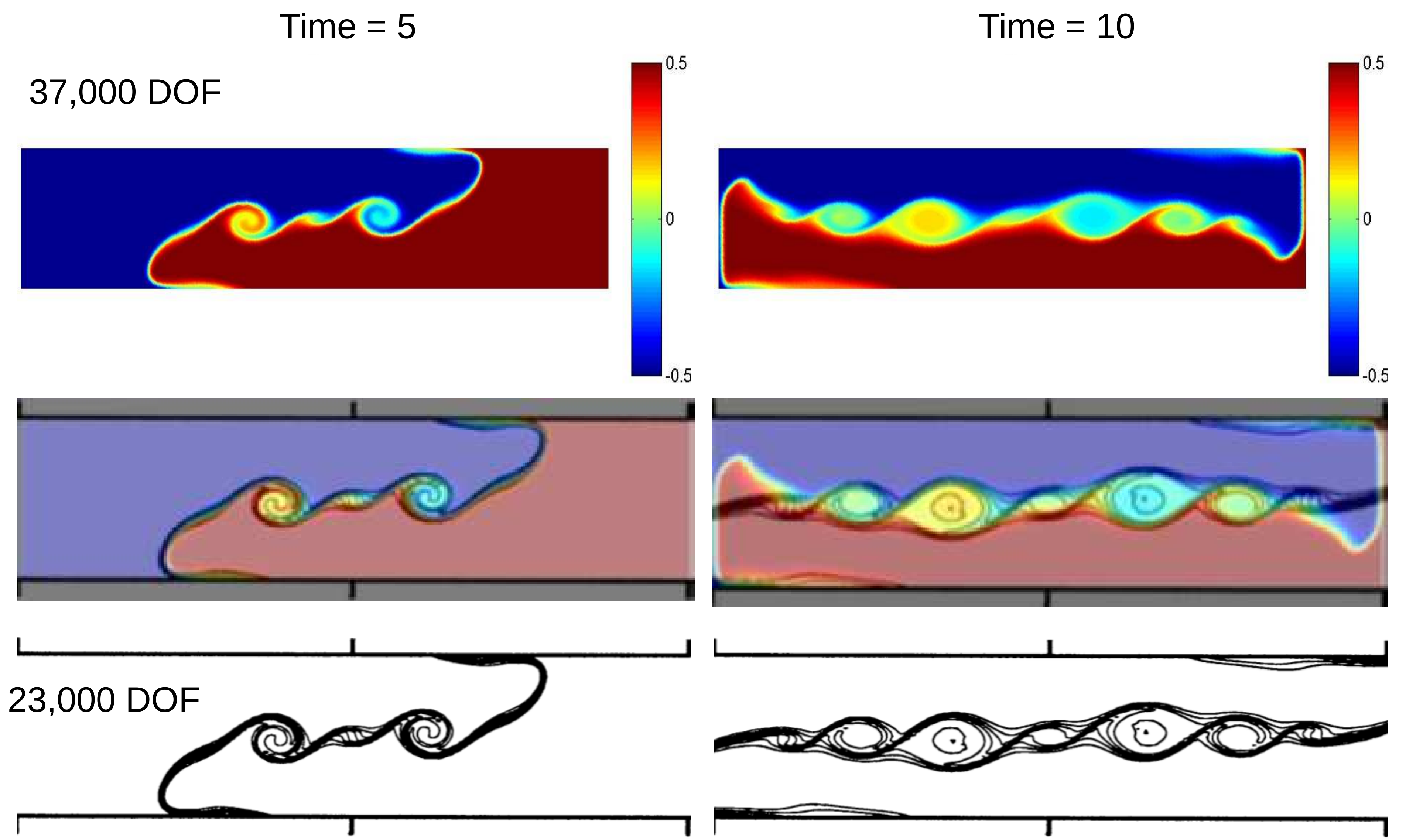






Hartel, C., Meinburg, E., and Freider, N. (2000). *Analysis and direct numerical simulations of the flow at a gravity-current head. Part 1. Flow topology and front speed for slip and no-slip boundaries.* J. Fluid. Mech, 418:189-212.

Lock Exchange Problem



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