Multigrid-based solvers for the shallow-water equations

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Fast iterative multigrid solvers in fluid mechanics have been the subject of a large amount of research since the 1980's [Barros *et al.*, 1990; Benzi *et al.*, 2005; Elman *et al.*, 2005; Saad *et al.*, 1986]. In ocean modeling, the development of numerical techniques is important to improve the implicit treatment of the stiff dynamics. For example, an efficient iterative solver for the computation of the sea surface could allow for getting rid of time step constraints imposed by gravity waves. The computational cost would be highly reduced, using a multi-scale numerical approach. Then, the ocean global circulation could be simulated by finite element models such as the *Second-generation Louvain-la-Neuve Ice-ocean Model* (SLIM) [White *et al.*, 2008]. Simulating a three-dimensional stratified flow using unstructured grids with finite element methods requires the development of new accurate and stable discrete formulations [Blaise *et al.*, 2010; Comblen *et al.*, 2010] waves or advective transport), while finite element methods were first developed for problems dominated by elliptic terms. Unstructured grid marine modeling is an active area of research for coastal applications (e.g. [Deleersnijder *et al.*, 2008]). Indeed, the coastlines must be accurately represented, as they have a much stronger influence at the regional scale than at the global scale.

The three-dimensional baroclinic free-surface marine SLIM model relies on a hydrostatic Boussinesq equation discretized with a Discontinuous Galerkin method on a mesh of prisms extruded in several layers from an unstructured two-dimensional mesh of triangles. As the prisms are vertically aligned, the calculation of the vertical velocity and the baroclinic pressure gradient can be implemented in an efficient and accurate way. All discrete fields are defined in discontinuous finite element spaces, in order to take advantage of the well-known good properties of the Discontinuous Galerkin methods for advection dominated problems and for wave problems. The discretization of the three-dimensional horizontal momentum and the continuity equations are defined in such a way that their discrete integration along the vertical axis would degenerate in a stable discrete formulation of the shallow-water equations. We stabilize the discrete equations by using the exact Riemann solver of the linear shallow-water equations for the gravity waves. Such a Riemann solver for the two-dimensional equations can be then viewed as a quite good approximate Riemann solver for the three-dimensional baroclinic equations. For internal waves, an additional stabilizing term is derived from a Lax-Friedrich solver. Consistency is ensured [White *et al.*, 2008]. The model is able to advect exactly a tracer with a constant concentration, meaning that the discrete transport term is compatible with the continuity equation.

As a first step in the development of an efficient preconditioned Krylov solver of the three-dimensional baroclinic ocean model, we propose to design an efficient multigrid-based solver for the two-dimensional shallow-water equations

$$\begin{aligned}\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} + f \underline{e}_z \times \underline{u} &= -g \nabla \eta + \frac{1}{H} \nabla \cdot (H \nu_t \nabla \underline{u}) + \frac{\underline{\tau}^w}{\rho H}, \\ \frac{\partial \eta}{\partial t} + \nabla \cdot (H \underline{u}) &= 0,\end{aligned}$$

where the unknown fields $\underline{u}$, and η are the velocity and the sea surface elevation, respectively. The latter can be viewed as a simple translation of the total depth $H = \eta + h$ where h is the depth at rest. The Coriolis factor, the gravitational acceleration and the density are denoted by f, g and ρ, respectively. The wind forcing is provided by $\underline{\tau}^w$ and the dissipation is ensured through a turbulent viscosity ν_t. To obtain a numerical solution, a linearization of the equations is usually performed. In order to ensure the convergence of the Newton scheme, it is mandatory to introduce a pseudo-time stepping algorithm that will guide the iterative global scheme toward the solution while avoiding unphysical intermediate steps that could lead to numerical unstabilities. The selection of the pseudo time step can be critical for the convergence of both the preconditioned GMRES and the Newton schemes. The fact that the linearized system is approximately solved is also a key issue. To perform a mathematical analysis, let us assume a constant bathymetry h and remove the inertia terms and all nonlinear free surface terms. Then, the equations degenerate into an usual saddle-point problem [Benzi *et al.*, 2005] exhibiting a similar mathematical structure as the Stokes equations

$$\begin{aligned}-\nabla \cdot (\nu_t \nabla \underline{u}) + f \underline{e}_z \times \underline{u} + g \nabla \eta &= \frac{\underline{\tau}^w}{\rho h}, \\ \nabla \cdot \underline{u} &= 0,\end{aligned}$$

where the sea surface elevation η plays here the role of the pressure. The finite element discretization of this simplified model yields the linear equations

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ g\boldsymbol{\eta} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}$$

where A contains the discrete diffusion and Coriolis operator; B and B^T are the discrete divergence and gradient operators, respectively.

To solve such a discrete saddle-point problem with a rate of convergence independent of the mesh size, we use a block approach. An approximation of an ideal preconditioner based on the Schur complement is obtained from a geometric multigrid approach [Elman *et al.*, 2005; Hackbush, 1985]. We consider approximate L_2 projections between non-nested meshes and multi-directional smoothers using block Gauss–Seidel splittings Special care for the design of the preconditioner is required to take the impact of both the Coriolis force and the discontinuous discretization into account. Nevertheless, our approach is general and can then be applied on both standard and non-conforming Galerkin finite element methods.

References

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