Multirate Time Stepping for Accelerating Explicit High Order Discontinuous Galerkin Computations

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The development of suitable and fast time integration methods for ocean modeling constitutes an important challenge. No single time-discretisation works well for all physical processes in a complex marine model, as different subsystems have widely different characteristics in terms of time scales, dynamic behaviour, and accuracy requirements. The primitive equations for ocean flows allow for the existence of phenomena exhibiting a wide spectrum of propagation speeds. Typically, external gravity waves propagate at $10^1 - 10^2\ ms^{-1}$ and internal waves at a few meters per second, whereas advection is characterized by speeds ranging from 10^{-3} to $1\ ms^{-1}$.

We believe that building appropriate time stepping strategies for multi-scale computations will enable us to gain an order of magnitude. For instance, consider the case of a typical mesh of the Great Barrier Reef (GBR) made up of about 1 million triangles. Element sizes were determined in order to capture the relevant bathymetric and topographic features, and the associated hydrodynamic processes, such as eddies and tidal jets (Lambrechts *et al.*, 2008). Unstructured-mesh generation processes are complex and, even though it is possible to control average element sizes in specific regions of the domain, it is not the case for each element size. The smallest element is usually much more smaller than the criterion that was prescribed a priori. Typically, it is often possible that more than 99% of the elements have a size that is much more larger than the smallest element. So, if we apply a local Courant–Friedrichs–Lewy (CFL) condition, the global time step would be critically smaller than the one required for 99% of the elements.

We propose a class of methods that use various time steps on different grid

cells. It elegantly addresses varying cell sizes between adapted elements. For multirate schemes, the strategy consists in splitting the domain in a smart way. Grid cells are gathered in different multirate groups that satisfy the local CFL stability conditions for a certain range of time steps. Standard Explicit Runge-Kutta (ERK) methods are applied, on independent partitions, with a local time step in such a way that the total computational efforts are drastically reduced. Finally, we ensure that these different time steps are globally synchronized in order to be consistent.

Whereas the multirate approach considerably attempts to reduce the computational costs of explicit time integration by adapting the characteristic time steps locally, the development of such methods is still challenging. Both, stability requirements and conservation properties should be satisfied. The major difficulty is intrinsically linked with the interface treatment between the different multirate partitions. In this context, we explore two approaches that should accommodate these transitions with adapted methods. The first one, introduced by Constantinescu *et al.*, 2007, preserves the system invariants but is at most second order accurate. On the contrary, Schlegel *et al.*, 2009, proposed a method that is based on a right-hand side splitting and borrows some ideas of the implicit-explicit splitting scheme. Unfortunately this method turns out to be non-conservative. It is proved that a third order multirate scheme can be achieved with an appropriate base ERK method.

Applications like the GBR require the use of parallel computers. Some kind of load balancing strategy has to be supplied to accomodate multirate time stepping schemes: indeed, small elements have a higher coast than large elements in such a strategy. Moreover, small elements at inter-processor interfaces will require more frequent updates. The key idea consists in creating an optimized mesh partition in a way that the amount of grid cells of the different multirate groups is ideally the same on each computer core. In other words we require that each processor treats a partition where the grid cell's spectrum is equivalent. This problem can be posed as a graph partitioning problem with multiple objectives.

References

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