

1 **A GEOMETRIC APPROACH TO**
2 **DYNAMICAL MODEL-ORDER REDUCTION**

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4 **Abstract.** Any model order reduced dynamical system that evolves a modal decomposition to
5 approximate the discretized solution of a stochastic PDE can be related to a vector field tangent
6 to the manifold of fixed rank matrices. The Dynamically Orthogonal (DO) approximation is the
7 canonical reduced order model for which the corresponding vector field is the orthogonal projection
8 of the original system dynamics onto the tangent spaces of this manifold. The embedded geometry of
9 the fixed rank matrix manifold is thoroughly analyzed. The curvature of the manifold is characterized
10 and related to the smallest singular value through the study of the Weingarten map. Differentiability
11 results for the orthogonal projection onto embedded manifolds are reviewed and used to derive an
12 explicit dynamical system for tracking the truncated Singular Value Decomposition (SVD) of a
13 time-dependent matrix. It is demonstrated that the error made by the DO approximation remains
14 controlled under the minimal condition that the original solution stays close to the low rank manifold,
15 which translates into an explicit dependence of this error on the gap between singular values. The
16 DO approximation is also justified as the dynamical system that applies instantaneously the SVD
17 truncation to optimally constrain the rank of the reduced solution. Riemannian matrix optimization
18 is investigated in this extrinsic framework to provide algorithms that adaptively update the best low
19 rank approximation of a smoothly varying matrix. The related gradient flow provides a dynamical
20 system that converges to the truncated SVD of an input matrix for almost every initial data.

21 **Key words.** Model order reduction, fixed rank matrix manifold, low rank approximation,
22 Singular Value Decomposition, orthogonal projection, curvature, Weingarten map, Dynamically Or-
23 thogonal approximation, Riemannian matrix optimization.

24 **AMS subject classifications.** 65C20, 53B21, 65F30, 15A23, 53A07, 35R60, 65M15

25 **1. Introduction.** Finding efficient model order reduction methods is an issue
26 commonly encountered in a wide variety of domains involving intensive computa-
27 tions and expensive high-fidelity simulations [66, 61, 38, 13]. Such domains include
28 uncertainty quantification [25, 46, 69, 72], dynamical systems analysis [30, 9, 80], elec-
29 trical engineering [24, 8], mechanical engineering [54], ocean and weather predictions
30 [41, 49, 12, 62], chemistry [55], and biology [40], to name a few. The computational
31 costs and challenges arise from the complexity of the mathematical models as well
32 as from the needs of representing variations of parameter values and the dominant
33 uncertainties involved. For example, to quantify uncertainties of dynamical system
34 fields, one often needs to solve stochastic partial differential equations (PDEs),

35 (1)
$$\partial_t \mathbf{u} = \mathcal{L}(t, \mathbf{u}; \omega),$$

36 where t is time, \mathbf{u} the uncertain dynamical fields, \mathcal{L} a differential operator, and
37 ω a random event. For deterministic but parametric dynamical systems, ω may
38 represent a large set of possible parameter values that need to be accounted for by
39 the model-order reduction. Generally, after both spatial and stochastic/parametric
40 event discretization of the PDE (1), or more directly if the focus is on solving a
41 complex system of ordinary differential equations (ODEs), one is interested in the
42 numerical solution of a large system of ODEs of the form

43 (2)
$$\dot{\mathfrak{R}} = \mathcal{L}(t, \mathfrak{R}),$$

44 where \mathcal{L} is an operator acting on the space of l -by- m matrices \mathfrak{R} . In the case of
45 a direct Monte-Carlo approach for the resolution of the stochastic PDE (1), \mathcal{L} is

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