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DYNAMICALLY ORTHOGONAL NUMERICAL SCHEMES FOR EFFICIENT STOCHASTIC ADVECTION AND LAGRANGIAN TRANSPORT

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Abstract. Quantifying the uncertainty of Lagrangian motion can be performed by solving a large number of ordinary differential equations with random velocities, or equivalently a stochastic transport partial differential equation (PDE) for the ensemble of flow-maps. The Dynamically Orthogonal (DO) decomposition is applied as an efficient dynamical model order reduction to solve for such stochastic advection and Lagrangian transport. Its interpretation as the method that applies instantaneously the truncated SVD on the matrix discretization of the original stochastic PDE is used to obtain new numerical schemes. Fully linear, explicit central advection schemes stabilized with numerical filters are selected to ensure efficiency, accuracy, stability, and direct consistency between the original deterministic and stochastic DO advections and flow-maps. Various strategies are presented for selecting a time-stepping that accounts for the curvature of the fixed rank manifold and the error related to closely singular coefficient matrices. Efficient schemes are developed to dynamically evolve the rank of the reduced solution and to ensure the orthogonality of the basis matrix while preserving its smooth evolution over time. Finally, the new schemes are applied to quantify the uncertain Lagrangian motions of a 2D double gyre flow with random frequency and of a stochastic flow past a cylinder.

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1. Introduction.

Advection plays a major role in a wide variety of physical processes and engineering applications of fluid mechanics [26, 3], neutronic transport, chemical transports, atmospheric sciences [62] and ocean sciences [20, 53]. At its most fundamental level, the pure advection process is commonly understood through the transport partial differential equation (PDE),

$$\begin{cases} (\partial_t + \boldsymbol{v}(t,\boldsymbol{x})\cdot\nabla)\boldsymbol{\psi} = 0 \\ \boldsymbol{\psi}(0,\boldsymbol{x}) = \boldsymbol{\psi}_0(\boldsymbol{x}), \end{cases} \tag{1}$$

that models the material transport of a passive (scalar or vectorial) tracer field $\boldsymbol{\psi}$ under a velocity field $\boldsymbol{v}$, having initially its values distributed as $\boldsymbol{\psi}_0$ over a physical domain $\Omega \subset \mathbb{R}^d$ of positions $\boldsymbol{x}$. Another description of transport considers a parcel of material initially located at the location $\boldsymbol{x}_0$ and transported to the position $\boldsymbol{\phi}_0^t(\boldsymbol{x}_0) = \boldsymbol{x}(t)$ with the instantaneous velocity $\boldsymbol{v}(t,\boldsymbol{x}(t))$. In this Lagrangian description, $\boldsymbol{x}(t)$ is the solution of the ordinary differential equation (ODE)

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{v}(t,\boldsymbol{x}(t)) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0, \end{cases} \tag{2}$$

and $\boldsymbol{\phi}_0^t$, *i.e.* the function mapping the initial positions $\boldsymbol{x}_0$ to those $\boldsymbol{\phi}_0^t(\boldsymbol{x}_0) = \boldsymbol{x}(t)$ at time t, is the *flow-map* of the ODE (2). Under sufficient regularity conditions on the velocity field $\boldsymbol{v}$ [9, 2], the solution $\boldsymbol{\psi}$ of the advection eq. (1) relates to (2) as being obtained by "carrying $\boldsymbol{\psi}_0$ values along particles' paths":

$$\boldsymbol{\psi}(t,\boldsymbol{x}) = \boldsymbol{\psi}_0((\boldsymbol{\phi}_0^t)^{-1}(\boldsymbol{x})), \tag{3}$$

where $(\boldsymbol{\phi}_0^t)^{-1}$ is the backward or inverse flow-map (Figure 1).

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