

1 **DYNAMICALLY ORTHOGONAL NUMERICAL SCHEMES**
2 **FOR EFFICIENT STOCHASTIC ADVECTION**
3 **AND LAGRANGIAN TRANSPORT**

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5 **Abstract.** Quantifying the uncertainty of Lagrangian motion can be performed by solving a
6 large number of ordinary differential equations with random velocities, or equivalently a stochastic
7 transport partial differential equation (PDE) for the ensemble of flow-maps. The Dynamically Or-
8 thogonal (DO) decomposition is applied as an efficient dynamical model order reduction to solve for
9 such stochastic advection and Lagrangian transport. Its interpretation as the method that applies
10 instantaneously the truncated SVD on the matrix discretization of the original stochastic PDE is
11 used to obtain new numerical schemes. Fully linear, explicit central advection schemes stabilized
12 with numerical filters are selected to ensure efficiency, accuracy, stability, and direct consistency
13 between the original deterministic and stochastic DO advections and flow-maps. Various strategies
14 are presented for selecting a time-stepping that accounts for the curvature of the fixed rank manifold
15 and the error related to closely singular coefficient matrices. Efficient schemes are developed to dy-
16 namically evolve the rank of the reduced solution and to ensure the orthogonality of the basis matrix
17 while preserving its smooth evolution over time. Finally, the new schemes are applied to quantify the
18 uncertain Lagrangian motions of a 2D double gyre flow with random frequency and of a stochastic
19 flow past a cylinder.

20 **AMS subject classifications.** 65C20, 53B21, 15A23, 35R60

21 **1. Introduction.** Advection plays a major role in a wide variety of physical
22 processes and engineering applications of fluid mechanics [26, 3], neutronic transport,
23 chemical transports, atmospheric sciences [62] and ocean sciences [20, 53]. At its most
24 fundamental level, the pure advection process is commonly understood through the
25 transport partial differential equation (PDE),

26 (1)
$$\begin{cases} (\partial_t + \mathbf{v}(t, \mathbf{x}) \cdot \nabla)\psi = 0 \\ \psi(0, \mathbf{x}) = \psi_0(\mathbf{x}), \end{cases}$$

27 that models the material transport of a passive (scalar or vectorial) tracer field ψ
28 under a velocity field \mathbf{v} , having initially its values distributed as ψ_0 over a physical
29 domain $\Omega \subset \mathbb{R}^d$ of positions \mathbf{x} . Another description of transport considers a parcel of
30 material initially located at the location \mathbf{x}_0 and transported to the position $\phi_0^t(\mathbf{x}_0) =$
31 $\mathbf{x}(t)$ with the instantaneous velocity $\mathbf{v}(t, \mathbf{x}(t))$. In this Lagrangian description, $\mathbf{x}(t)$
32 is the solution of the ordinary differential equation (ODE)

33 (2)
$$\begin{cases} \dot{\mathbf{x}} = \mathbf{v}(t, \mathbf{x}(t)) \\ \mathbf{x}(0) = \mathbf{x}_0, \end{cases}$$

34 and ϕ_0^t , *i.e.* the function mapping the initial positions \mathbf{x}_0 to those $\phi_0^t(\mathbf{x}_0) = \mathbf{x}(t)$ at
35 time t , is the *flow-map* of the ODE (2). Under sufficient regularity conditions on the
36 velocity field \mathbf{v} [9, 2], the solution ψ of the advection eq. (1) relates to (2) as being
37 obtained by “carrying ψ_0 values along particles’ paths”:

38 (3)
$$\psi(t, \mathbf{x}) = \psi_0((\phi_0^t)^{-1}(\mathbf{x})),$$

39 where $(\phi_0^t)^{-1}$ is the backward or inverse flow-map (Figure 1).

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