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THE EXTRINSIC GEOMETRY OF CONTINUOUS TIME MATRIX ALGORITHMS*

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Abstract. A geometric framework is introduced for the systematic study of a class of maps called oblique projections. These include *orthogonal* projections, *i.e.* maps that project points of a finite dimensional ambient vector space to the closest point of an embedded manifold, but also their generalizations to non euclidean ambient spaces. Typical everyday examples involve the truncated SVD, the polar decomposition, linear subspace filters, and projectors over the dominant eigenspaces of symmetric and non symmetric matrices. A methodology is proposed for the systematic derivation of the differential of these maps, and of convergent continuous time matrix algorithms that allow to dynamically track their values on time dependent matrices. It is shown that these maps are characterized by a bundle of normal spaces that provide the image manifold with a differentiable structure. Generalizations of classical properties of embedded Riemannian manifolds, such as the Gauss equation and the Weingarten identity, are found by replacing the ambient scalar product with the duality bracket. Previous differentiability results obtained for orthogonal projections onto embedded Riemannian manifolds are extended to oblique projections in the non euclidean setting. The framework is applied to the study of the maps above and their image manifold in an embedded setting, that include the Stiefel and the Orthogonal group, the Isospectral and the Grassman manifold, and the bi-Grassman manifold or the set of fixed rank linear projectors.

Key words. Oblique and orthogonal projections; Stiefel manifold; Polar Decomposition; dynamic dominant eigenspaces; Isospectral, Grassman and bi-Grassman manifolds; normal bundle.

AMS subject classifications. 65C20, 53B21, 65F30, 15A23, 53A07

1. Introduction.

Continuous time matrix algorithms have been receiving a growing interest in a wide range of applications including data assimilation [25], data processing [42], machine learning [18] and matrix completion [41]. A large number of works has focused on deriving dynamical systems that, given an input matrix $\mathfrak{R}$, compute an algebraic operation, such as eigenvalue, singular value, or polar decomposition [12, 7, 11, 13, 20]. One of the most popular examples is probably the double-bracket flow introduced by Brockett in 1988 [7]:

$$\dot{H} = [H, [H, \mathbf{S}]] = H(H\mathbf{S} - \mathbf{S}H) - (H\mathbf{S} - \mathbf{S}H)H,$$

which, given an initial symmetric matrix $H(0)$, aligns the eigenvectors of the solution $H(t)$ to those of a symmetric input matrix $\mathbf{S}$. These dynamical systems may not be numerically competitive for a single computation of the matrix operation when compared to direct algebraic algorithms. Nevertheless, they become extremely beneficial when integrated in time marching procedures requiring constant updates of matrix decompositions. Such situations can also occur when the input matrices themselves are governed by high-dimensional dynamical systems.

Generally, one often considers a given time dependent matrix $\mathfrak{R}(t)$, whose trajectory may (or may not) be prescribed by an Ordinary Differential Equation (ODE)

$$\dot{\mathfrak{R}} = \mathcal{L}(t, \mathfrak{R}),$$

and one is interested in tracking the value of an algebraic operation $\Pi_{\mathscr{M}}(\mathfrak{R}(t))$ for which repeated evaluations of direct algorithms at every time step would be expensive or intractable. Typical examples of maps $\Pi_{\mathscr{M}}$ that are considered in this paper include:

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