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Sparse and Deep Gaussian Process Closure Modeling for Non-Stationary Two-Dimensional β-Plane Vorticity Flows Past Idealized Obstacles

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High-resolution simulations that fully resolve all spatio-temporal scales of geophysical and turbulent flows remain a challenge in large ocean domains [1, 2]. Large-eddy simulations (LES) make these computations tractable by filtering out subgrid-scale (SGS) features, but require accurate closures to remain stable and faithful; without them, solutions can drift, lose energy at the wrong rate, or develop spurious coastal artifacts [3–5]. Classical analytical closures based on the eddy-viscosity hypothesis, such as the Smagorinsky and Leith models and their dynamic variants [6–8], were developed primarily for three-dimensional homogeneous turbulence. A recent benchmarking study [9] has shown that they logically only weakly capture the SGS forcing in two-dimensional vorticity flows in the presence of coastal boundaries and interior landforms, motivating the development of data-driven closures. Among such approaches, neural-network closures [10–15] have shown promise but typically return only a deterministic point estimate of the SGS term, while the mapping from resolved to unresolved scales is fundamentally non-invertible and the closure is therefore intrinsically stochastic [16–21]. This non-uniqueness becomes especially pronounced in non-stationary flows, where the wake statistics themselves drift in time and a single deterministic correction can likely not represent the spread of admissible SGS responses.

In this work, we develop and evaluate sparse and deep Gaussian process (GP) closures for under-resolved, non-stationary two-dimensional classical and β-plane vorticity flows past idealized obstacles, where non-stationarity is driven by a time-modulated inflow velocity $U_\infty(t)$ that produces a continuously evolving wake, with shedding frequency, wake width, and subgrid-scale statistics all drifting along the trajectory. Gaussian processes are well suited to closure modeling in fluids: they encode smoothness and invariance through kernels, learn nonparametric mappings from data, and return calibrated posterior uncertainty alongside the mean correction [22]. To model the SGS forcing, we adopt a hybrid architecture in which a shallow convolutional neural network (CNN) extracts local features from the resolved fields, motivated by the finding that the SGS stresses in two-dimensional and β-plane turbulence depend only weakly non-locally on the coarse fields and are therefore well represented by small CNNs [20, 23, 24]; the extracted features are then mapped to the closure source term by a sparse variational GP head with inducing points [25, 26]. GP regression has been used directly as the predictive component of data-driven turbulence closures: Ho et al. [27] learn an ensemble of per-case GP emulators that map local flow features to a field-inversion-derived correction for the k–ω SST model, and exploit the ensemble's predictive variance to improve generalization to unseen geometries; Zighed et al. [28] use GP regression for super-resolution, recovering fine-resolution fields from filtered ones in a hybrid VAE-transformer-GP framework for Kolmogorov flow. Here the GP head plays a different role: it directly predicts the SGS closure source term in the filtered vorticity equation from CNN-extracted resolved-field features, with the deep-kernel-learning construction [29] combining the representational efficiency of shallow CNNs for such flows with the predictive accuracy and calibrated posterior uncertainty of GP regression. The closure is embedded intrusively in the pseudo-spectral solver of [9] and evaluated online at every coarse step, keeping it consistent with the numerics and providing a per-step uncertainty estimate. Rather than training on a single trajectory, we train on an *ensemble* of fully resolved realizations driven by different modulation functions $U_\infty(t)$ (varying in amplitude, frequency, phase, and waveform), so that the GP posterior naturally encodes the spread of SGS responses consistent with a family of non-stationary inflow forcings.

We consider the two-dimensional vorticity-streamfunction (ω, ψ) formulation on the β-plane,

$$\begin{aligned} \frac{\partial \omega}{\partial t} + J(\psi, \omega) &= \nu \nabla^2 \omega - \mu \omega - \beta \frac{\partial \psi}{\partial x} + F + F_{\text{obs}}, \\ (u, v) &= \left(-\tfrac{\partial \psi}{\partial y},\, \tfrac{\partial \psi}{\partial x}\right), \quad \omega = \nabla^2 \psi, \end{aligned} \tag{1}$$

where $J(\psi, \omega)$ is the nonlinear Jacobian, ν the kinematic viscosity, μ the linear bottom-drag coefficient, and β the meridional gradient of the Coriolis parameter. The planetary-vorticity-gradient term $\beta\, \partial_x \psi$ supports Rossby waves and zonal jets and provides the geophysical character of the flow. Idealized obstacles (circular islands, coastal capes, and submarine-like bluff bodies) are imposed without mesh modification through a Brinkman volume-penalization forcing F_{obs} [30, 31], and the inflow velocity upstream is prescribed as a time-dependent signal $U_\infty(t)$. The boundary-generated wakes are inherently non-stationary, with transient vortex shedding.

We assess the proposed sparse and deep GP closures on three obstacle-induced transient wake regimes: (i) flow past an idealized circular island, (ii) flow past a coastal cape with several β's, and (iii) flow past an idealized submarine. For each obstacle, we generate an ensemble of fully resolved reference simulations driven by a family of modulation functions $\{U_\infty^{(i)}(t)\}_{i=1}^{N_{\text{ens}}}$ that vary in amplitude, frequency, phase, and waveform. Models are trained on filtered downsamplings of a subset of these realizations and evaluated online at coarser resolutions on *held-out modulation functions* that were not seen during training, providing a clean test of out-of-distribution generalization across non-stationary forcing regimes within the same physical setup. Performance is quantified using a comprehensive suite of skill metrics: pointwise L_2 and L_∞ errors of the resolved fields, a-priori and a-posteriori Pearson correlation of the SGS forcing Π and the vorticity ω with their filtered references [9], total kinetic-energy and enstrophy spectra, time-resolved wake-integrated lift and drag coefficients $C_L(t)$, $C_D(t)$, and instantaneous Strouhal number $\mathrm{St}(t)$. We additionally evaluate the calibration of the GP posterior uncertainty by comparing predictive intervals against the empirical SGS error along each trajectory, and assess whether the posterior widens appropriately on out-of-distribution modulation regimes.

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