Saravanakumar, A.K. and P.F.J. Lermusiaux, 2026. A Hybridizable Discontinuous Galerkin Solver for Quasi-Geostrophic Flows in Complex Domains. In: OCEANS '26 IEEE/MTS Monterey, 21–24 September 2026, sub-judice.

A Hybridizable Discontinuous Galerkin Solver for Quasi-Geostrophic Flows in Complex Domains

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***Index Terms*—ocean modeling, high-order, quasi-geostrophic flow, unstructured grid, hybridizable discontinuous Galerkin**

Numerical modeling of ocean dynamics is critical for studying and predicting a wide range of geophysical phenomena, including mesoscale turbulence [1] and coastal circulation. However, resolving the wide range of spatial scales present in such flows, particularly in domains with complex coastlines and bathymetry, remains computationally challenging. Low-order finite difference and finite volume schemes often require fine spatial resolution to capture turbulent structures, leading to intractable computational costs [2, 3]. Pseudo-spectral solvers [4, 5] provide an attractive alternative due to their exponential accuracy and computational efficiency for smooth solutions, but they are generally restricted to simple periodic geometries or require special boundary schemes [5]. High-order discontinuous Galerkin finite element methods [6] offer a suitable compromise by combining geometric flexibility with high-order accuracy, enabling computationally tractable simulations of turbulent flows in complex domains.

In this work, we implement a high-order hybridizable discontinuous Galerkin method-based (HDG) finite element solver for the two-dimensional quasi-geostrophic (QG) ocean equations [7, 8]. The vorticity-streamfunction (ω, ψ) formulation of the QG equations is,

$$\frac{\partial \omega}{\partial t} + J(\psi, \omega) = \frac{1}{\mathrm{Re}}\nabla^2\omega - \mu\omega - \beta\frac{\partial \psi}{\partial x} + F, \qquad (1)$$
$$\nabla^2\psi = \omega$$

where, $J(\psi, \omega)$ is the nonlinear Jacobian operator, Re is the Reynolds number, μ is the linear bottom drag coefficient, and β is the Rossby parameter (meridional gradient of the Coriolis parameter). While standard discontinuous Galerkin (DG) methods [9] are well-suited for complex geometries, they incur increased computational costs due to duplicated degrees of freedom across element interfaces. The HDG framework [10–15] alleviates this overhead by introducing hybrid trace variables and performing static condensation, thereby significantly reducing the number of globally coupled unknowns. The globally coupled degrees of freedom associated with different DG schemes are shown in Fig. 1. We present a HDG-discretization of the vorticity-streamfunction formulation of the QG equations, with emphasis on consistent and stable numerical fluxes. We discuss the treatment of boundary conditions [16–18] which is crucial for accurately representing boundary effects such as shear generation and vorticity production near coastlines. The unstructured grid capabilities of the solver allow for targeted resolution of viscous boundary dynamics near coastlines. For example, the Munk layer thickness δ_M [19, 20] which characterizes lateral boundary layer scale in wind-driven gyres, can be locally resolved with mesh refinement. We also note that the QG equations reduce to the 2D incompressible Navier-Stokes equations in the limit of $\mu = \beta = 0$, enabling simulation of classical 2D flows.

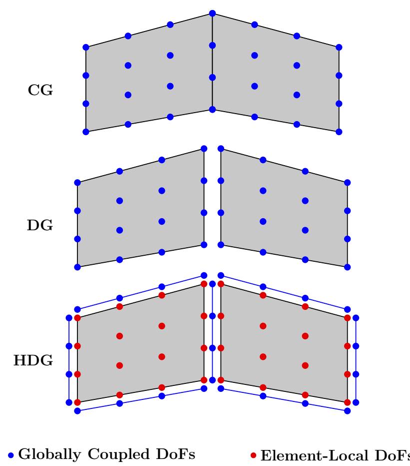

Fig. 1. Degrees of freedom (DoFs) associated with the i) continuous Galerkin (CG), ii) standard discontinuous Galerkin (DG), and iii) hybridizable discontinuous Galerkin (HDG) finite element methods.

We validate our solver using a doubly periodic forced turbulence test case, where we demonstrate good agreement with benchmark results in terms of energy and enstrophy evolution. We then showcase the ability of the method to handle complex coastal geometries by simulating quasi-geostrophic flow in domains with irregular boundaries, including flows in domains containing coastal features such as capes and islands [21, 22]. As an example of this capability, Fig 2 shows forced turbulence in a domain inspired by the Alboran Sea region [23].

Finally, we present a performance analysis of the solver, highlighting the computational advantages of static condensation and discussing scalability considerations for large-scale simulations. The combination of high-order accuracy, reduced global system size, and geometric flexibility positions the proposed HDG QG solver as an efficient framework for mesoscale coastal ocean modeling application. We also discuss future extensions toward GPU-accelerated implementations [24] and simulations on realistic ocean geometries.

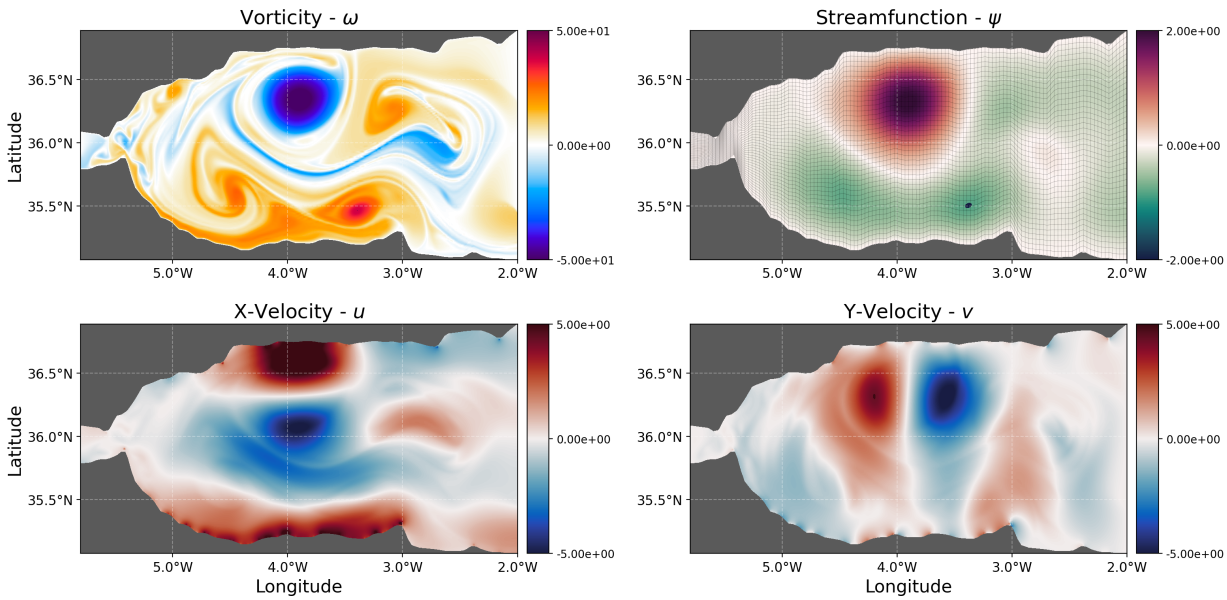

Fig. 2. Simulation of forced quasi-geostrophic turbulence in an idealized domain inspired by the Alboran Sea. The fields shown are the vorticity ω, streamfunction ψ, and velocity components (u, v). The unstructured quadrilateral mesh used for the simulation is overlaid on the streamfunction contour.

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